

معادلات تفاضلية

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إرادة - ثقة - تغيير

Def. An equation containing the derivative of one or more dependent variables with respect to one or more independent variables, is said to be differentiable equation "DE".

Ex:

$$1. y'' + 2y' = 3x$$

$$2. \frac{dy}{dx} + 2xy = e^x$$

$$3. \frac{\partial y}{\partial x} + \frac{\partial y}{\partial z} = 0$$

classification:

→ differentiable equations are classified by:

II Type

ordinary DE

→ containing the ordinary derivative.

$$\text{Ex: } 1. \frac{dy}{dx} + 3y = \ln x$$

$$2. y'' + y' = 3y$$

partial DE

→ containing the partial derivative

$$\text{Ex: } 1. U_{xx} - U_{yy} = 0$$

$$2. \frac{\partial y}{\partial x} + \frac{\partial y}{\partial z} = 3xz$$

[2] order :

→ The order of a DE is the order of the highest derivative in the eq.

Ex :-

$$1. y'' + 2x(y')^3 = e^x \quad \therefore \text{order} = 5$$

$$2. y' \cdot y''' + y^{(5)} + 2x = 0 \quad \therefore \text{order} = 5$$

$$3. (u_x)^4 + \frac{\partial^2 u}{\partial x \partial y} + uy = 0 \quad \therefore \text{order} = 2$$

$$4. y^3 \left[\frac{dy}{dx} (y'' \cdot y^{(3)}) + 2xy^{(3)} \right] = 5 \quad \therefore \text{order} = 4$$

[3] classification as linear or non-linear :

→ The linear ODE of n^{th} order has the form,

$$a_n(x) \frac{dy^n}{dx^n} + a_{n-1}(x) \frac{dy^{n-1}}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x)y = y^n$$

Ex :- classify the DE as linear or non-linear.

$$1. y'' + 3xy' + 5e^x y = \sin x \quad \bullet \text{ linear}$$

$$2. y^{(5)} + 3xy \cdot y' + 5x = 0 \quad \bullet \text{ non linear}$$

$$3. \frac{dy}{dx} + \frac{d^3 y}{dx^3} + \ln(xy) = e^x \quad \bullet \text{ non linear}$$

*** Remark :**

The general form of ODE with n^{th} order is:

$$\frac{d^n y}{dx^n} = F(x, y, y', y'', \dots, y^{(n-1)})$$

• Solution :

Explicit solution

implicit solution

[solution is a function]

[Relation between

$$y = F(x)$$

x and y]

$$\text{Ex: } x^2 + y^2 = 25$$

Ex:

Show that $y = \phi(x) = x^2 - x^{-1}$ is an explicit solution for

$$y'' - \frac{2}{x^2} y = 0$$

$$y = x^2 - x^{-1}$$

$$y' = 2x + x^{-2}$$

$$y'' = 2 - 2x^{-3}$$

$$y'' - \frac{2}{x^2} y = 2 - 2x^{-3} - \frac{2}{x^2} (x^2 - x^{-1})$$

$$= 2 - 2x^{-3} - 2 + 2x^{-3} = 0 \quad \checkmark$$

Subject _____ Day _____ Date _____

Ex:

Show that $4x^2 - y^2 = 4$ is an implicit solution
to $y \frac{dy}{dx} - 4x = 0$

$$4x^2 - y^2 = 4$$

$$(8x - 2y \frac{dy}{dx} = 0) \cdot -\frac{1}{2}$$

$$+\frac{1}{2} \cdot 2y \frac{dy}{dx} - \frac{8x}{2} = 0 \rightarrow y \frac{dy}{dx} - 4x = 0 \quad \checkmark$$

Initial value problem (I.V.P)

'solve the following I.V.P'

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$

$$\text{Where } y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

Ch. 2 :

First order ordinary Differential Equations:

this DE has the form :

$$y' = f(x, y)$$

• Separable Equation :

This equation has the form

$$y' = f(x) \cdot g(y)$$

Ex :

$$1. y' = yx + y$$

This DE is separable because

$$y' = y(x+1)$$

$\nwarrow$
 $g(y)$

$\downarrow$
 $f(x)$

2. $y' = e^{2x+y}$

separable because $y' = e^{2x} \cdot e^y$

3. $y' = \sin(x+y) \therefore$ not separable.

To solve the separable Equation

$$y' = f(x) \cdot g(y)$$

$$\rightarrow \frac{dy}{dx} = f(x) \cdot g(y)$$

$$\frac{1}{g(y)} dy = f(x) dx \quad \text{and Integrate both sides}$$

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

Ex: solve

$$\frac{dy}{dx} = x^2 y + y \rightarrow \frac{dy}{dx} = (x^2 + 1) y \quad \text{separable}$$

$$\int \frac{1}{y} dy = \int (x^2 + 1) dx$$

$$\ln|y| = \frac{x^3}{3} + x + C$$

$$|y| = e^{\frac{x^3}{3} + x} \cdot e^C \rightarrow \text{let 'A'}$$

$$|y| = A e^{\frac{x^3}{3} + x} \rightarrow y = \pm A e^{\frac{x^3}{3} + x}$$

• solution

Ex: Solve

$$x e^{-y} \sin x \, dx - y \, dy = 0$$

$$y \, dy = x e^{-y} \sin x \, dx$$

$$\int y e^y \, dy = \int x \sin x \, dx \quad \text{"separable"}$$

$$y e^y - e^y = -x \cos x + \sin x + C$$

$$e^y (y - 1) = -x \cos x + \sin x + C$$

↪ implicit solution.

$\begin{array}{r} \int \\ \oplus y \\ \ominus 1 \\ 0 \end{array} \begin{array}{l} \searrow \\ \searrow \end{array} \begin{array}{c} e^y \\ e^y \\ e^y \end{array}$		$\begin{array}{r} \int \\ \oplus x \\ \ominus 1 \\ 0 \end{array} \begin{array}{l} \searrow \\ \searrow \end{array} \begin{array}{c} \sin x \\ -\cos x \\ -\sin x \end{array}$
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Ex:

Solve the following I.V.P.

$$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$$

$\therefore$ where $y(0) = 3$

$$\frac{dy}{dx} = \frac{x(y+3) - (y+3)}{x(y-2) + 4(y-2)}$$

$$\frac{dy}{dx} = \frac{(y+3)(x-1)}{(y-2)(x+4)}$$

$$\frac{dy}{dx} = \frac{y+3}{y-2} \cdot \frac{x-1}{x+4} \quad \text{* separable *$$

$$\int \frac{y-2}{y+3} dy = \int \frac{x-1}{x+4} dx \Rightarrow \begin{array}{l} \frac{1}{y-2} \overline{) y+3} \\ \underline{y+3} \\ -5 \end{array} \quad \begin{array}{l} \frac{1}{x-1} \overline{) x+4} \\ \underline{x-1} \\ 5 \end{array}$$

$$\Rightarrow \int \left(1 - \frac{5}{y+3}\right) dy = \int \left(1 - \frac{5}{x+4}\right) dx$$

$$y - 5 \ln |y+3| = x - 5 \ln |x+4| + C$$

To find $C \Rightarrow y(0) = 3 \rightarrow y$

$$* 3 - 5 \ln |3+3| = 0 - 5 \ln |0+4| + C \rightarrow 3 - 5 \ln 6 = -5 \ln 4 + C$$

$$C = 3 + 5 \ln 4 - 5 \ln 6 = 3 + 5 (\ln 4 - \ln 6) \rightarrow$$

→

$$= 3 + 5 \left(\ln \frac{y}{3} \right) = 3 + 5 \left(\ln \frac{2}{3} \right)$$

The solution is:

$$y - 5 \ln |y+3| = x - 5 \ln |x+4| + 3 + 5 \ln \frac{2}{3}$$

* H.W *

• solve:

$$1. \frac{dy}{dx} = \frac{xy + 2y - x - 2}{xy - 3y + x - 3}$$

$$2. \frac{dy}{dt} = e^{\ln(y) - t} \quad \text{see } y(1+t^2)$$

$$3. (1+x) dy - y dx = 0$$

4. Find the value of m such that the following DE is separable

$$\frac{dy}{dx} = \frac{xy - y + 3x - 3}{xy - 2x + 4y + m}$$

$$\underline{\text{Ans}} : \frac{dy}{dx} = \frac{y(x-1) + 3(x-1)}{x(y-1) + 4(y + \frac{m}{4})} \rightarrow -2 \leq \frac{m}{4}$$

$m \leq -8$

Linear Equation

The standard form of this Eq is:

$$y' + p(x)y = f(x) \quad \text{--- (A)}$$

To solve this equation

1. Find the Integrating Factor

$$M(x) = \int p(x) dx$$

2. Multiply the eq (A) by $M(x)$

$$[M(x) y]' + p(x) M(x) y = M(x) f(x)$$

$$[M(x) y]' = M(x) f(x)$$

3. Integrate both sides with respect to x

$$\int [M(x) y]' dx = \int M(x) f(x) dx$$

$$M(x) y = \int M(x) f(x) dx + C$$

solution:

$$y = \frac{1}{M(x)} \left(\int M(x) f(x) dx + C \right)$$


Since :

$$[M(x)y]' = M(x)y' + M'(x)y$$

$$= M(x)y' + p(x)M(x)y$$

$$\begin{aligned} M'(x) &= e^{\int p(x)dx} \\ &= p(x) e^{\int p(x)dx} \\ &= p(x) M(x) \end{aligned}$$

Linear Equation

The standard form is

$$y' + p(x)y = f(x)$$

* The integrating factor is

$$M(x) = e^{\int p(x) dx}$$

* The solution is

$$y = \frac{1}{M(x)} \left(\int M(x) f(x) dx + C \right)$$

Ex:

so we:

$$xy' - 4y = x^6 e^x$$

"linear"

$$(*) \quad y' + \underbrace{\left(-\frac{4}{x}\right)}_{p(x)} y = \underbrace{x^5 e^x}_{f(x)}$$

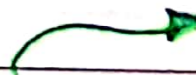
* The integrating factor is

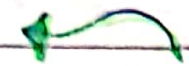
$$M(x) = e^{\int p(x) dx} = e^{\int -\frac{4}{x} dx}$$

$$= e^{-4 \ln x}$$

$$= e^{\ln x^{-4}}$$

$$= x^{-4}$$





⊗ The solution is:

$$y = \frac{1}{M(x)} \left(\int M(x) P(x) dx + C \right)$$

$$y = \frac{1}{x^{-4}} \int x^{-4} \cdot x^5 e^x dx + C$$

$$= x^4 \left(\int \underbrace{x e^x}_{\text{اجزاء}} dx + C \right)$$

$$= x^4 (x e^x - e^x + C)$$

$$y = x^5 e^x - x^4 e^x + C x^4$$

	$\int$
⊕ x	e^x
⊖ 1	e^x
⊙ 0	e^x

Ex:

Solve.

$$(x+2)^2 \frac{dy}{dx} = 5 - 4y - 2xy \quad \text{"linear"}$$

$$(x+2)^2 y' + 4y + 2xy = 5$$

$$(x+2)^2 y' + 2y(2+x) = 5$$



قسم

$$(x+2)^2 \Rightarrow y'' + 2 \frac{(2+x)}{(x+2)^2} y = \frac{5}{(x+2)^2}$$

$$y' + \underbrace{\frac{2}{x+2}}_{P(x)} y = \underbrace{\frac{5}{(x+2)^2}}_{f(x)}$$

The integrating factor is:

$$\begin{aligned} \mu(x) &= e^{\int P(x) dx} = e^{\int \frac{2}{x+2} dx} \\ &= e^{2 \ln(x+2)} \\ &= e^{\ln(x+2)^2} \\ &= (x+2)^2 \end{aligned}$$

- the solution is

$$y = \frac{1}{\mu(x)} \left(\int \mu(x) f(x) dx + C \right)$$

$$= \frac{1}{(x+2)^2} \left(\int (x+2)^2 \cdot \frac{5}{(x+2)^2} dx + C \right)$$

$$= \frac{1}{(x+2)^2} \cdot (5x + C) = (x+2)^{-2} \cdot 5x + C$$

$$= \frac{5x + C}{(x+2)^2}$$

H.W.

solve

$$\frac{dr}{d\theta} + r \sec \theta = \cos \theta$$

Ex.

If $M(x) = 2^{\ln(x)}$ is an integrating factor for linear equation $xy' + 3p(x)y = f(x)$

find the function $p(x)$.

* Remark:

$$p(x) = \frac{M'(x)}{M(x)}$$

$$xy' + 3p(x)y = f(x)$$

$$y' + \frac{3p(x)}{x}y = \frac{f(x)}{x}$$

y delete
stander 3

y delete

$$M(x) = e^{\int \frac{3p(x)}{x} dx}$$

$$2^{\ln x} = e^{\int \frac{3p(x)}{x} dx}$$

$$\rightarrow \ln(2^{\ln x}) = \int \frac{3p(x)}{x} dx$$

$$\ln x \cdot \ln 2 = \int \frac{3p(x)}{x} dx \rightarrow \frac{\ln 2}{x} = \frac{3p(x)}{x} \text{ diff}$$

$$p(x) = \frac{\ln 2}{3}$$

* by Remark

$$\frac{3p(x)}{x} = \frac{M'}{M} = \frac{\frac{1}{x} \cdot 2^{\ln x} \ln 2}{2^{\ln x}}$$

$$= \frac{3p(x)}{x} = \frac{\ln 2}{x} \Rightarrow p(x) = \frac{\ln 2}{3}$$

Bernoulli's Equation:

The standard form of this equation is:

$$y' + q(x)y = g(x)y^n, \text{ for all } n \in \mathbb{R}$$

Note that :-

1. If $n=0$, then the above equation becomes:

$$y' + q(x)y = g(x) \quad \text{"linear"}$$

2. If $n=1$, then the above eq.

becomes, $y' + q(x)y = g(x)y$

$$y' + (q(x) - g(x))y = 0 \quad \text{"linear"}$$

3. For $n \neq 0$ and $n \neq 1$, the substitution $v = y^{1-n}$ transforms the Bernoulli eq into linear as:

1- Dividing that eq ~~by~~ on y^n then we have:

$$\frac{1}{y^n} (y' + q(x)y = g(x)y^n)$$

$$\frac{y'}{y^n} + \frac{q(x)y}{y^n} = g(x) \Rightarrow \frac{y'}{y^n} + q(x)y^{1-n} = g(x)$$

2- since $v = y^{1-n}$

$$v' = (1-n)y^{-n}y' \rightarrow \frac{v'}{(1-n)} = y^{-n}y'$$

put this in eq (1):

$$\frac{y'}{y^n} + q(x) y^{1-n} = g(x) \rightarrow \tilde{y}^{-n} \cdot y' + q(x) y^{1-n} = g(x)$$

$$\frac{v'}{1-n} + q(x)v = g(x)$$

$$v' + (1-n)q(x)v = (1-n)g(x) \quad \text{"linear in v"}$$

$$* (y')^{1-n} + q(x)y = g(x)(y)^n \rightarrow \text{Bernolli}$$

$$v = y^{1-n}$$

$$v' + (1-n)q(x)v = (1-n)g(x)$$

Ex 2 solve

$$y' - 5y = \frac{-5}{2} x y^3 \quad \text{Bernolli}$$

$$v' + 10v = 5x \quad \text{linear in v} \quad n=3 \quad v = y^{1-n} = y^{-2}$$

* integrating factor is:

$$\mu(x) = e^{\int p(x) dx} = e^{\int 10 dx} = e^{10x}$$

$$* v = \frac{1}{\mu(x)} \left(\int \mu(x) f(x) dx + C \right)$$



$$y^{-2} = v = \frac{1}{e^{10x}} \left(\int 5x e^{10x} dx + C \right)$$

$$y^{-2} = e^{10x} \left(\frac{5x e^{10x}}{10} - \frac{5}{100} e^{10x} + C \right)$$

$$y^{-2} = \frac{x}{2} - \frac{1}{20} + C e^{-10x}$$

$$\Rightarrow \frac{1}{y^2} = \frac{x}{2} - \frac{1}{20} + C e^{-10x}$$

$$\Rightarrow y^2 = \frac{1}{\frac{x}{2} - \frac{1}{20} + C e^{-10x}}$$

$$\begin{array}{l} e^{5x} \rightarrow e^{10x} \\ \ominus 5 \rightarrow \frac{e^{10x}}{10} \\ 0 \rightarrow \frac{e^{10x}}{100} \end{array}$$

$$y = \sqrt{\frac{1}{\frac{x}{2} - \frac{1}{20} + C e^{-10x}}} = \frac{1}{\sqrt{\frac{x}{2} - \frac{1}{20} + C e^{-10x}}}$$

Ex:

solve

$$x \frac{dy}{dx} + y = \ln x y^2 \rightarrow \text{Bernoulli'}$$

$$x y' + y = \ln x y^2 \Rightarrow y' + \frac{y}{x} = \frac{\ln x}{x} y^2$$

$$= v' + \frac{1}{x} v = \frac{\ln x}{x} v^2 \quad v = \frac{-\ln x}{x} \quad \text{linear}$$



$$M(x) = e^{\int P(x) dx} = e^{\int \frac{1}{x} dx} = e^{-\ln x} = x^{-1} = \frac{1}{x}$$

$$* V = \frac{1}{M(x)} \left(\int u(x) Q(x) dx + C \right)$$

$$y^{-1} = \frac{1}{x-1} \left(\int \frac{-\ln x}{x^2} dx + C \right)$$

$$\rightarrow y^{-1} = \frac{1}{x-1} \left(\int \frac{-\ln x}{x^2} dx + C \right)$$

$$\rightarrow y^{-1} = \frac{1}{x-1} \left(x^{-1} \ln x - \int x^{-1} \frac{1}{x} dx \right)$$

$$\rightarrow y^{-1} = \frac{1}{x-1} \left(x^{-1} \ln x - \int x^{-2} dx + C \right)$$

$$\rightarrow = \frac{1}{x-1} \left(x^{-1} \ln x - \int x^{-2} dx + C \right)$$

$$\rightarrow = \ln x + 1 + C x$$

$$\rightarrow y = \frac{1}{\ln x + 1 + C x}$$

Ex.

Solve

$$\frac{dy}{dx} = \frac{1}{\frac{x}{y} + yx^2} \quad \begin{matrix} * n=2 & 1-n \\ v=x & =x^{-1} \end{matrix}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} + yx^2$$

$$x' - \frac{1}{y} \cdot x = yx^2 \quad \text{"Bernoulli"}$$

$$v' + \underbrace{\left(-\frac{1}{y}\right)}_{p(y)} v = y \rightarrow \text{linear.}$$

$$\mu(y) = e^{\int p(y) dy} = e^{\int -\frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$$

$$* v = \frac{1}{\mu(y)} \left(\int \mu(y) f(y) dy + C \right)$$

$$x^{-1} = v = \frac{1}{y} \left(\int y \cdot (-y) dy + C \right)$$

$$x^{-1} = \frac{1}{y} \left(-\frac{y^3}{3} + C \right) \Rightarrow x^{-1} = \frac{-y^2}{3} + \frac{C}{y}$$

$$\frac{-y^3 + C}{3y} \Rightarrow \sqrt{\frac{-3y}{-y^2 + C}} = x$$

H.W.

Solve

1. $y' + xy = 5x\sqrt{y}$

2. $y' + xy = xy^2$

Exact Equation:

The first order DE

$$M(x, y) dx + N(x, y) dy = 0$$

is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ " $M_y = N_x$ "

Ex 3

Is the following DE exact?

$$\boxed{1} \quad \underbrace{(2xy^2 - 3xy + 4y - x + 1)}_M dx + \underbrace{(2x^2y - \frac{3}{2}x^2 + 2y + 4)}_N dy$$

$$M(x, y) = 2xy^2 - 3xy + 4y - x + 1$$

$$\frac{\partial M}{\partial y} = M_y = 4xy - 3x + 4 - 0 + 0$$

$$= 4xy - 3x + 4$$



$$\textcircled{*} N(x, y) = 2x^2 y - \frac{3}{2} x^2 + 2y + 4x$$

$$\frac{\partial N}{\partial x} = N_x = 4xy - 3x + 0 + 4$$

$$= 4xy - 3x + 4$$

$$M_y = N_x \Rightarrow \text{exact.}$$

$$\textcolor{red}{2.} \quad \underbrace{(ye^{xy} + 2x)}_M dx + \underbrace{(xe^{xy} - 2y)}_N dy = 0$$

$$* M(x, y) = ye^{xy} + 2x$$

$$M_y = 1 \cdot e^{xy} + y \cdot xe^{xy} + 0 = e^{xy} + ye^{xy}$$

$$* N(x, y) = xe^{xy} - 2y$$

$$N_x = 1 \cdot e^{xy} + x \cdot ye^{xy} - 0$$

$$= e^{xy} + xye^{xy}$$

$$- M_y = N_x \Rightarrow \text{exact.}$$

$$\textcolor{red}{3.} \quad \underbrace{\sin(x+2y)}_N dy - \underbrace{\cos(x+2y)}_M dx = 0$$

$$M(x, y) = -\cos(x+2y)$$

$$M_y = \sin(x+2y) \cdot (0+2)$$

$$= 2 \sin(x+2y)$$

• $N(x, y) = \cos(x + 2y)$

$$N_x = -\sin(x + 2y) \cdot (1 + 0)$$

$$= -\sin(x + 2y)$$

$$M_y \neq N_x \Rightarrow \text{Not exact.}$$

* To solve the Exact Equation $M(x, y)dx + N(x, y)dy = 0$ ($M_y = N_x$)

suppose that the solution is :

$$P(x, y) = C \quad \text{where}$$

$$\frac{\partial P}{\partial x} = M \quad \text{and} \quad \frac{\partial P}{\partial y} = N$$

Ex:

solve

$$\underbrace{(2xy + 3x - 1)}_M dx + \underbrace{(x^2 - 2y + 5)}_N dy = 0$$

$$M_y = 2x, \quad N_x = 2x$$

$$M_y = N_x \Rightarrow \text{exact.}$$

suppose the solution is :

$$P(x, y) = C \quad \text{where}$$

$\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$

$\therefore \frac{\partial F}{\partial x} = 2xy + 3x - 1$, $\frac{\partial F}{\partial y} = 2x^2 - 2y + 5$

$\therefore \int \frac{\partial F}{\partial x} dx = \int (2xy + 3x - 1) dx$, $\frac{x^2 + 0 - 0 + g'(y)}{x^2 - 2y + 5} = \dots$

$\therefore f(x, y) = x^2y + \frac{3}{2}x^2 - x + g(y)$, $\downarrow$
 $x^2 + g'(y) = x^2 - 2y + 5$
 $g'(y) = -2y + 5$

$\int g'(y) dy = \int -2y + 5 dy$
 $\rightarrow g(y) = -y^2 + 5y$

$\therefore f(x, y) = x^2y + \frac{3}{2}x^2 - x - y^2 + 5y$

The solution is $f(x, y) = C$

$x^2y + \frac{3}{2}x^2 - x - y^2 + 5y = C$

Remark: $Mdx + Ndy = 0$ exact.

then the solution is

$\int M(x, y) dx + \int \frac{\partial M}{\partial y} dy = C$



OR $\int M dx + \int N dy = C$

م $\int$ $\frac{dx}{dy}$

Ex:

Solve

$$(2xy + 3x - 1) dx + (x^2 - 2y + 5) dy = 0$$

M N

$$M_y = 2x$$

$$N_x = 2x$$

$$M_y = N_x \Rightarrow \text{exact.}$$

the solution is :

$$\int (2xy + 3x - 1) dx + \int (-2y + 5) dy = C$$

$$\rightarrow x^2y + \frac{3}{2}x^2 - x - y^2 + 5y = C$$

Ex:

Find the general solution.

$$\underbrace{(\cos x \sin x - xy^2)}_M dx + \underbrace{(y - yx^2)}_N dy = 0$$

$$M_y = 0 - 2xy = -2xy$$

$$N_x = 0 - 2xy$$

$$M_y = N_x \Rightarrow \text{exact.}$$

$$= -2xy$$

The solution is:

$$\int (\cos x \sin x - xy^2) dx + \int y dy = c$$

$$\Rightarrow \int \frac{1}{2} \sin(2x) - xy^2 dx + \int y dy = c$$

$$= -\frac{\cos(2x)}{4} - \frac{1}{2} x^2 y^2 + \frac{y^2}{2} = c$$

H.W.

soln:

$$(ye^{xy} + y^2 - \frac{y}{x^2} + x) dx + (xe^{xy} + 2xy + \frac{1}{x}) dy = 0$$

Ex 3

Find the value of a and b such that

$$\underbrace{(xy^3 + by^2)}_M dx + \underbrace{(axy^2 + xy)}_N dy = 0$$

is exact:

sol: since the DE is exact $\Rightarrow My = Nx$

$$3xy^2 + 2by = 2axy^2 + xy$$

$$\therefore 2a = 3 \Rightarrow a = \frac{3}{2}$$

$$\text{and } 2b = 1 \Rightarrow b = \frac{1}{2}$$

Remark:

If $M dx + N dy = 0$ is exact equation
then

$$1. M(x, y) = \int N dx + h(y)$$

$$2. N(x, y) = \int M dy + g(x)$$

Ex:

Find the most general function $M(x, y)$ so that the following DE is exact.

$$\textcircled{1} M(x, y) dx + (\sec^2 y - \frac{x}{y}) dy = 0$$

⇒ since DE exact then,

$$M_y = N_x$$

$$M_y = -\frac{1}{y}$$

$$\int M_y dy = \int -\frac{1}{y} dy + h(x)$$

$$= M(x, y) = -\ln(y) + h(x) \quad \leftarrow \text{function}$$

Remember:

$$M(x, y) = -\ln|y| \quad \checkmark$$

$$M(x, y) = -\ln|y| + 3 \quad \checkmark$$

$$M(x, y) = -\ln|y| + K \quad \checkmark$$

$$M(x, y) = -\ln|y| + xy + K \quad \times$$

H.W.

$$(2) M(x, y) dx + (\sin x \cos y - xy - e^{-y}) dy = 0$$

Ex:

Find the function $N(x, y)$ such that the following DE is exact.

$$1. (y \cos(xy) + e^x) dx + N(x, y) dy = 0$$

$$\text{where } N(x, y) = 2xy$$

Sol: Since the DE is exact, $\Rightarrow N_x = M_y$

$$N_x = 1 \cos(xy) + y \cdot -\sin x \cdot x + 0$$

$$\int N_x = \cos(xy) - xy \sin(xy)$$

$$\rightarrow N(x, y) = \int \cos(xy) - xy \sin(xy) dx + g(y)$$

$$N(x, y) = \int \cos(xy) dx - \int xy \sin(xy) dx + g(y)$$

$$\textcircled{*} N(x, y) = \frac{\sin(xy)}{y} - \left[\frac{-xy \cos(xy)}{y} + \frac{y \sin(xy)}{y} \right] + g(y)$$



$x \rightarrow r$

$\int dx$

$\oplus xy$

$\sin(xy)$

$\ominus y$

$-\cos(xy)$

$\oplus$

0

$\frac{\sin(xy)}{y^2}$

$$\Rightarrow N(x, y) = \frac{\sin(xy)}{y} + x \cos(xy) - \frac{\sin(xy)}{y} + g(y)$$

$$N(x, y) = x \cos(xy) + g(y)$$

• since $N(1, y) = 2y$

$$\Rightarrow 1 \cdot \cos(1 \cdot y) + g(y) = 2y$$

$$\cos(y) + g(y) = 2y$$

$$g(y) = 2y - \cos y$$

$$N(x, y) = x \cos(xy) + 2y - \cos y$$

'H.W':

$$(y e^{xy} - 4x^3 y + 2) dx + N(x, y) dy = 0$$

where $N(2, y) = e^y - 1$

Remark:

If $f(x, y) = c$ is a solution for exact equation. $M dx + N dy = 0$

$$\text{then: } M(x, y) = \frac{\partial f}{\partial x}, \quad N(x, y) = \frac{\partial f}{\partial y}$$

Ex:

If $\cos(xy) - 2x + \ln y = e^x - 3y^2 + 5$ is a solution for exact e.g.

$$x \cdot M(x, y) dx + 2 N(x, y) dy = 0$$

Find M and N :

$$\rightarrow \cos(xy) - 2x + \ln y - e^x + 3y^2 - 5 = 0$$

$$f(x, y)$$

$$\textcircled{*} \quad x M(x, y) = \frac{\partial f}{\partial x} \Rightarrow -y \sin(xy) - 2 - e^x$$

$$\bullet \quad M(x, y) = \frac{-y \sin(xy) - 2 - e^x}{x}$$

$$2 N(x, y) = \frac{\partial f}{\partial y} \Rightarrow 2 N(x, y) = -\sin(xy) + \frac{1}{y} + 6y$$

$$\bullet \quad N(x, y) = -\frac{1}{2} \sin(xy) + \frac{1}{2y} + 3y$$

Special Integrating Factor:

For the non-exact DE

$$Mdx + Ndy = 0 \quad (M_y \neq N_x)$$

we have:

1. If $\frac{M_y - N_x}{N}$ depends on x only then the

special integrating factor is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

then the DE becomes exact if we multiply the DE by $\mu(x)$.

2. If $\frac{N_x - M_y}{M}$ depends on y only then the special integrating factor is:

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

if multiply the DE by $\mu(y)$, then DE becomes exact.

Ex:

solve the I.V.P

$$\underbrace{y dx}_M + \underbrace{(2x - ye^y) dy}_N = 0, \quad y(0) = 1$$

$$\because My = 1, \quad Nx = 2 \Rightarrow My \neq Nx \quad \text{not exact.}$$

$$\frac{Nx - My}{M} = \frac{2 - 1}{y} = \frac{1}{y} \rightarrow \text{يعتمد على } y \text{ فقط}$$

$$\textcircled{*} \mu(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

⊛ Multiply the DE by $\mu(y) = y$

$$\rightarrow y [y dx + (2x - ye^y) dy] = 0$$

$$y^2 dx + (2xy - y^2 e^y) dy = 0 \quad (My = Nx) \text{ exact.}$$

$$\text{the solution is } \int y^2 dx + \int -y^2 e^y dy = C$$

$$\boxed{x \frac{y^2}{2} + [-\frac{y^2}{2} + 2y - 2]e^y = C}$$



		$\int$
ⓐ	$-y^2$	e^y
ⓑ	$-2y$	e^y
ⓒ	-2	e^y
	0	e^y

since $y(0) = 1$

$$\rightarrow 0 \cdot 1^2 + [-11^2 + 2(1) - 2] e^1 = C$$

$\therefore$ the solution for I.V.P is :

$$xy^2 + [-y^2 + 2y - 2] e^y = -e$$

Ex 3

Solve the DE $\underbrace{(3x^2 + y)}_M dx + \underbrace{(xy^2 - x)}_N dy = 0$

$$M_y = 1, N_x = 2xy - 1$$

$M_y \neq N_x$ Not exact

$$\begin{aligned} \textcircled{A} \quad \frac{M_y - N_x}{N} &= \frac{1 - 2xy + 1}{x^2y - x} = \frac{2 - 2xy}{x(xy - 1)} \\ &= \frac{-2(xy - 1)}{x(xy - 1)} \end{aligned}$$

$$\text{Integrating} \leftarrow = \frac{-2}{x}$$

$$\hookrightarrow M(x) = e^{\int \frac{-2}{x} dx} = e^{-2 \ln x} = x^{-2}$$

⊗ Multiply the DE by $M(x) = x^{-2}$

$$\Rightarrow x^{-2} [(3x^2 + y)dx + (x^2 y - x)dy] = 0$$

$$(3 + yx^{-2})dx + (y - x^{-1})dy = 0 \quad \text{"exact"}$$

The g.s.

$$\rightarrow \int (3 + yx^{-2})dx + \int y dy = C$$

$$\left| 3x - yx^{-1} + \frac{y^2}{2} = C \right|$$

Ex:

If $M(x) = y^k$ is an integrating factor that makes the DE $xy dx + (2x^2 + 3y^2 - 2y) dy = 0$ is exact find the value of k .

solution:

Multiply the DE by $M(x) = y^k$

$$\Rightarrow \underbrace{xy^{k+1}}_M dx + \underbrace{(2x^2y^k + 3y^{k+2} - 2xy^k)}_N dy = 0 \quad \text{exact.}$$

$$M_y = N_x$$

$$(k+1)xy^k = 4xy^k \Rightarrow k+1=4 \Rightarrow k=3$$

Ex:

The ODE - $(x^2 + y)dx + P(x)dy = 0$

becomes exact by integrating factor $M(x)$

Find the function $P(x)$

Multiply the DE by $M(x) = x$

$$\underbrace{(x^3 + xy)}_M dx + \underbrace{xP(x)}_N dy = 0 \quad \text{exact.}$$

$$M_y = N_x$$

$$x = [xP(x)]_x$$

$$x = [xP(x)]'$$

$$\Rightarrow \int x dx = \int [xP(x)]' dx$$

$$\frac{x^2}{2} + C = xP(x)$$

$$P(x) = \frac{x}{2} + \frac{C}{x}$$

Remark:

If $M(x, y) = x^p \cdot y^q$ is an integrating factor
for DE $Mdx + Ndy = 0$, then to find p
and q we used the condition

$$My - Nx = p \cdot \frac{N}{x} - q \cdot \frac{M}{y}$$

Ex:

Find the integrating factor $M(x, y) = \rightarrow$
 $= x^p y^q$ for DE

$$\underbrace{(4xy^2 + 3y)}_M dx + \underbrace{(3x^2y + 2x)}_N dy = 0$$

$$= My - Nx = p \cdot \frac{N}{x} - q \cdot \frac{M}{y}$$

$$8xy + 3 - 6xy - 2 = p \cdot \frac{3x^2y + 2x}{x} - q \cdot \frac{4xy^2 + 3y}{y}$$

$$2xy + 1 = p(3xy + 2) - q(4xy + 3)$$

$$2y + 1 = 3pxy + 2p - 4qxy - 3q$$

$$\Rightarrow 2 = 3p - 4q \quad \text{--- (1)}$$

$$\Rightarrow 1 = 2p - 3q \quad \text{--- (2)}$$

$$\textcircled{2} \times \textcircled{1} - 3 \times \textcircled{2}$$



$$4 = 6p - 8q$$

$$+ \quad -3 = -6p + 9q$$

$$\boxed{1 = q}$$

put $q = 1$ in eq. (1) $\Rightarrow 2 = 3p - 4q$

$$2 = 3p - 4(1)$$

$$p = 2$$

(*) $M(x, y) = x^p \cdot y^q = x^2 y$

ex 18/67

If $xM(x, y) + yN(x, y) = 0$, find the solution to the equation.

$$M(x, y) dx + N(x, y) dy = 0$$

$$xM + yN = 0 \quad \Rightarrow \quad yN = -xM$$

$$\frac{y}{-x} = \frac{M}{N}$$

$$\rightarrow u dx + v dy = 0$$

$$v dy = -u dx \Rightarrow \frac{dy}{dx} = \frac{-u}{v} = \left(-\frac{M}{N} \right)$$

$$\frac{dy}{dx} = -\left(\frac{-y}{x} \right) = \frac{y}{x} \rightarrow \int \frac{1}{y} dy = \int \frac{1}{x} dx \quad \text{separable}$$

$$\ln |y| = \ln |x| + C \rightarrow |y| = C|x| \rightarrow y = \pm C|x|$$

Homogeneous Equation,

Def: A function $G(x, y)$ is said to be homogeneous of order n if $G(tx, ty) = t^n G(x, y)$

Ex:

Is $G(x, y) = x^2 + y^2$ homog of order 2??

yes because

$$G(tx, ty) = (tx)^2 + (ty)^2$$

$$= t^2 x^2 + t^2 y^2$$

$$= t^2 (x^2 + y^2) = t^2 G(x, y)$$

Ex:

Is $G(x, y) = x^2 + y^2 + xy$ homog?? No, because

$$G(tx, ty) = (tx)^2 + (ty)^2 + tx \cdot ty$$

$$= t^2 x^2 + t^2 y^2 + t^3 xy$$

$$= t^2 (x^2 + y^2 + txy)$$

The homog DE has the form:

$$\frac{dy}{dx} = h\left(\frac{y}{x}\right)$$

To solve this DE, we use the substitution

$$v = \frac{y}{x}, \text{ then the DE becomes separable}$$

Ex: Solve

$$\frac{dy}{dx} = \frac{4x^2 - xy}{5y^2 - 10x^2} \quad (\text{homog.})$$

$$\frac{dy}{dx} = \frac{x^2 \left(4 - \frac{y}{x}\right)}{x^2 \left(5 \frac{y^2}{x^2} - 10\right)}$$

$$\frac{dy}{dx} = \frac{4 - \frac{y}{x}}{5\left(\frac{y}{x}\right)^2 - 10}$$

$$v = \frac{y}{x} \rightarrow xv = y$$

$$v + x \frac{dv}{dx} = \frac{dy}{dx}$$

$$\rightarrow 5 + x \frac{dv}{dx} = \frac{4-v}{5v^2-10}$$

$$\frac{dv}{dx} = \frac{4-v}{5v^2-10} - v = \frac{4-v-5v^3+10v}{5v^2-10}$$

$$x \frac{dv}{dx} = \frac{4+9v-5v^3}{5v^2-10} \Rightarrow \frac{5v^2-10}{4+9v-5v^3} dv = \frac{1}{x} dx$$

$$\int \frac{5v^2-10}{4+9v-5v^3} dv = \ln|x| + C \quad \therefore \text{separable}$$



$$= \int \frac{5v^2 - 10}{5v^2 - 9v - 4} dv$$

$$v^2 - 1$$

$$= 5v^2 - 5v - 4 = 0$$

$$(\quad) (\quad)$$

$$5 \quad 0 \quad -9 \quad -4$$



$$\underline{\underline{-5 \quad 5 \quad 4}}$$

$$5 \quad -5 \quad -4 \quad 0$$

Ex:

Solve

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \quad (\text{homog})$$

$$\frac{dy}{dx} = \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$x + x \frac{dv}{dx} = 1 + v + v^2$$

$$x \frac{dv}{dx} = 1 + v^2$$

$$* v = \frac{y}{x}$$

$$xv = y$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\int \frac{1}{1+v^2} dv = \int \frac{1}{x} dx \quad * \text{separable}$$

$$\tan^{-1} v = \ln |x| + C$$

$$v = \tan(\ln |x| + C)$$

$$\frac{y}{x} = \tan(\ln |x| + C)$$

$$y = x \tan(\ln |x| + C)$$

Ex :

find the general solution for

$$\frac{dy}{dx} = \frac{2y + \sqrt{x^2 - y^2}}{2x} \quad (\text{homog})$$

$$= \frac{2y}{2x} + \frac{\sqrt{x^2 - y^2}}{2x}$$

$$= \frac{2y}{2x} + \frac{1}{2} \sqrt{\frac{x^2 - y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2} \sqrt{1 - \left(\frac{y}{x}\right)^2}$$

$$\rightarrow v + x \frac{dv}{dx} = v + \frac{1}{2} \sqrt{1 - v^2}$$

$$x \frac{dv}{dx} = \frac{1}{2} \sqrt{1 - v^2}$$

$$\int \frac{1}{\sqrt{1 - v^2}} dv = \int \frac{1}{2x} dx \quad \text{"separate"}'$$

$$\sin^{-1} v = \frac{1}{2} \ln|x| + C$$

$$v = \sin\left(\frac{1}{2} \ln|x| + C\right)$$

$$\frac{y}{x} = \sin\left(\frac{1}{2} \ln|x| + C\right)$$

$$y = x \sin\left(\frac{1}{2} \ln|x| + C\right)$$

$$* v = \frac{y}{x}$$

$$xv = y$$

$$v + x \frac{dv}{dx} = \frac{dy}{dx}$$

Ex:

Solve

$$(2\sqrt{xy} - y)dx - xdy = 0 \quad \text{'homog'}$$

$$x dy = (2\sqrt{xy} - y) dx$$

$$\frac{dy}{dx} = \frac{2\sqrt{xy} - y}{x} = \frac{2\sqrt{xy}}{x} - \frac{y}{x}$$

$$= 2\sqrt{\frac{y}{x}} - \frac{y}{x}$$

$$\Rightarrow v = \sqrt{\frac{y}{x}} \quad \frac{dv}{dx} = \frac{1}{2\sqrt{x}} \frac{dy}{dx} - \frac{y}{2x\sqrt{x}}$$

$$x \frac{dv}{dx} = 2\sqrt{v} - 2v = 2(\sqrt{v} - v)$$

$$\int \frac{1}{\sqrt{v} - v} dv = \int \frac{2}{x} dx \quad (\text{separable})$$

$$\int \frac{2w}{w - w^2} dw = 2 \ln|x| + C$$

$$\int \frac{2}{1 - w} dw = 2 \ln|x| + C$$

$$-2 \ln|1 - w| = 2 \ln|x| + C$$

$$-\ln|1 - \sqrt{y/x}| = \ln|x| + C$$

$$\ln|1 - \frac{y}{x}| = \ln|x| + C$$

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \\ v + x \frac{dv}{dx} &= \frac{dy}{dx} \end{aligned}$$

$$w = \sqrt{v}$$

$$w^2 = v$$

$$2w dw = dv$$

H.W.

Solve.

$$① \frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$$

$$② (x^2 e^{-\frac{y}{x}} + y^2) dx = xy dy$$

$$③ 2x^3 dy + (x^4 + y^4) dx = 0$$

* Equation of the form

$$\frac{dy}{dx} = f(cx + dy + e)$$

Equation of this form can always be reduced to an equation with separable by the substitution $v = cx + dy + e$.

Soln:

$$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$$

$$* \quad v = y - 2x + 3$$

$$v + 2x - 3 = y$$

$$\frac{dv}{dx} + 2 = 2 + \sqrt{v}$$

$$\frac{dv}{dx} + 2 = \frac{dy}{dx}$$

$$\frac{dv}{dx} = \sqrt{v}$$

$$\int \frac{1}{\sqrt{v}} dv = \int dx \quad \text{'separable'}$$

$$\int v^{-\frac{1}{2}} dv = x + C$$

$$2v^{\frac{1}{2}} = x + C \Rightarrow ((y - 2x + 3)^{\frac{1}{2}})^2 = (\frac{1}{2}x + C)^2$$

$$y - 2x + 3 = (\frac{1}{2}x + C)^2 \Rightarrow y = 2x - 3 + (\frac{1}{2}x + C)^2$$

~~$$* \quad y(x) = 2x - 3$$~~

Ex: solve

$$\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1}$$

$$\frac{dy}{dx} = y - x - 1 + (-(y - x) + 2)^{-1}$$

$$v = y - x$$

$$v + x = y$$

$$\frac{dv}{dx} + 1 = \frac{dy}{dx}$$

$$\frac{dv}{dx} + 1 = v - 1 + (-v + 2)^{-1}$$

$$\frac{dv}{dx} = v - 2 + \frac{1}{2 - v}$$

$$\frac{dv}{dx} = \frac{v - 2}{1} - \frac{1}{v - 2} = \frac{(v - 2)^2 - 1}{v - 2}$$

$$\Rightarrow \frac{v - 2}{(v - 2)^2 - 1} dv = \int dx \quad (\text{separable})$$

$$\frac{1}{2} \int \frac{2 * (v - 2)}{v^2 - 4v + 3} dv = x + C$$

$$\frac{1}{2} \ln |v^2 - 4v + 3| = x + C$$

$$\ln |v^2 - 4v + 3| = 2x + C$$

$$\ln |(y - x)^2 - 4(y - x) + 3| = 2x + C$$

Ex: solve

$$\frac{dy}{dx} = \sin(x-y)$$

$$* u = x-y$$

$$x-u = y$$

$$1 - \frac{du}{dx} = \frac{dy}{dx}$$

$$\Rightarrow 1 - \frac{du}{dx} = \sin u$$

$$-\frac{du}{dx} = \sin u - 1$$

$$\frac{du}{dx} = 1 - \sin u$$

$$\Rightarrow \int \frac{1}{1 - \sin u} du = \int dx \quad \text{'separable'}$$

$$\int \frac{1 + \sin u}{(1 - \sin u)(1 + \sin u)} du = x + c$$

$$\int \frac{1 + \sin u}{1 - \sin^2 u} du = x + c$$

$$\int \frac{1 + \sin u}{\cos^2 u} du = x + c$$

$$= \int \left(\frac{1}{\cos^2 u} + \frac{1}{\cos u} \cdot \frac{\sin u}{\cos u} \right) du = x + c$$

$$\int (\sec^2 u + \sec u \tan u) du = x + c$$

$$\rightarrow \tan u + \sec u = x + c \Rightarrow \tan(x-y) + \sec(x-y) = x + c$$

H.W.

Find the general solution to DE

$$y' = \left(\frac{1}{3x+y} \right)^5 + 3 \quad \text{where } 3x+y > 0$$

Theorem:

* Case linear | المعادلة الخطية *

consider the following I.V.P

$$y' + p(x)y = f(x) \text{ where } y(x_0) = y_0$$

If $p(x)$, $f(x)$ are cont function

on an open interval (a, b) contains x_0

* then there is a unique solution of I.V.P on that interval.

Ex:

Find the largest open interval in which a unique solution generated on exist.

$$\textcircled{1} (x^2-1) y' + \frac{\sin x}{\ln(x-3)} y = \frac{1}{x-5}, \quad y\left(\frac{7}{2}\right) = 10$$

soln:

$$\Rightarrow \underbrace{y' + \frac{\sin x}{(x^2-1)\ln(x-3)} y}_{p(x)} = \underbrace{\frac{1}{(x-5)(x^2-1)}}_{q(x)}$$

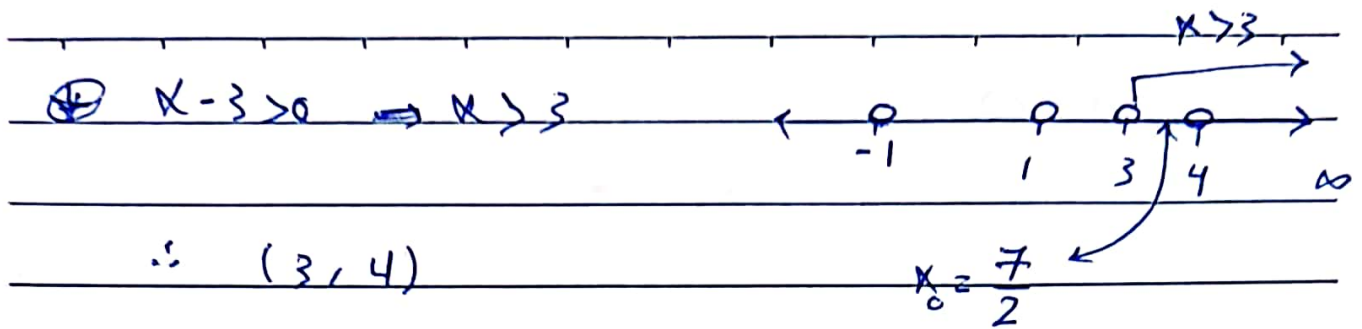
$$p(x) = \frac{\sin x}{(x^2-1)\ln(x-3)}$$

find the domain

$$(x^2-1)\ln(x-3) \neq 0 \Rightarrow x^2-1 \neq 0 \text{ or } \ln(x-3) \neq 0$$

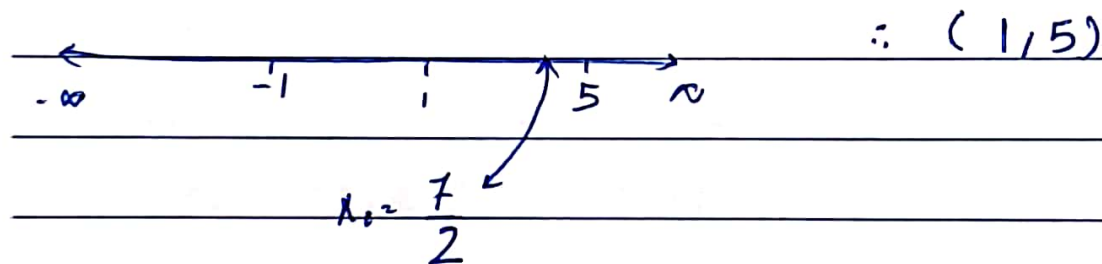
$$x \neq \pm 1 \quad x-3 \neq 1$$

$$x \neq 4$$



• $f(x) = \frac{1}{(x-5)(x^2-1)} \Rightarrow (x-5)(x^2-1) = 0$

$x = 5, x = \pm 1$



• The largest interval is:

$(3, 4) \cap (1, 5) = (3, 4)$

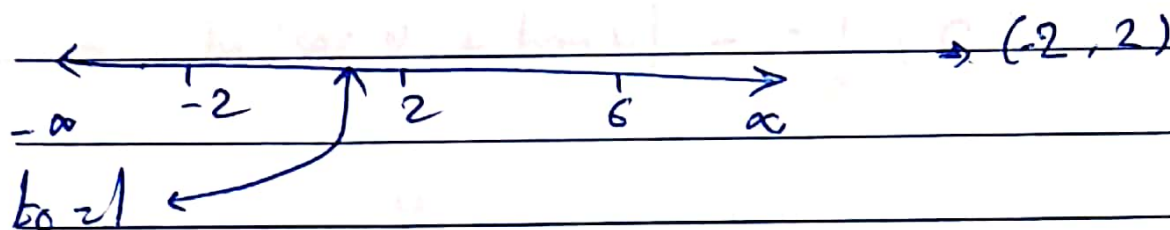
② $(4-t^2)y' + \ln(6-t)y = e^{-t}, y(1) = -9$

soln: $\hookrightarrow x_0 = 1$

$\Rightarrow y' + \frac{\ln(6-t)}{(4-t^2)}y = \frac{e^{-t}}{(4-t^2)}$

• $p(t) = \frac{\ln(6-t)}{(4-t^2)} \rightarrow 6-t=0 \rightarrow t < 6$

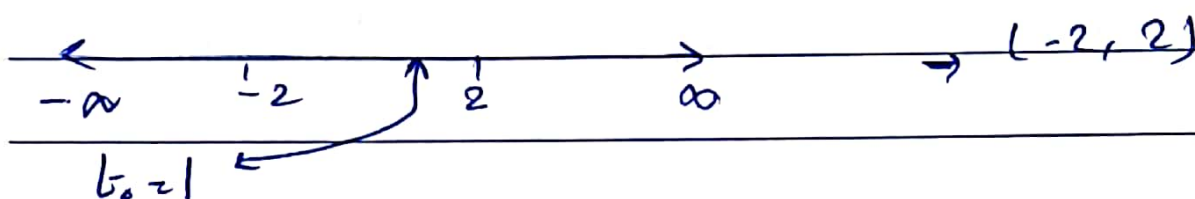
$q(t) = \frac{e^{-t}}{(4-t^2)} \rightarrow t \leq \pm 2$



$$P(t) = \frac{e^{-t}}{(4-t^2)}$$

$$4-t^2 \leq 0 \leftarrow \text{plot } t^2 \text{ and } 4$$

$$t = \pm 2$$



The largest interval is

$$(-2, 2) \cap (-2, 2) = (-2, 2)$$

Ex:

Find the general solution to the IDE:

$$\frac{dy}{dx} = \frac{2y}{x^2} + \cos\left(\frac{y}{x^2}\right)$$

solu:

$$2xv + x^2 \frac{dv}{dx} = 2xv + \cos v$$

$$\frac{1}{\cos v} dv = \frac{1}{x^2} dx \quad \text{separable}$$

$$v = \frac{y}{x^2}$$

$$x^2 v = y$$

$$2xv + x^2 \frac{dv}{dx} = \frac{dy}{dx}$$

$$xv = \frac{y}{x}$$

$$\int \frac{1}{\cos v} dv = \int \frac{1}{x^2} dx \Rightarrow \int \sec v dv = -\frac{1}{x} + C$$

$$\leftarrow \ln |\sec v + \tan v| = -\frac{1}{x} + C$$

$$\Rightarrow \ln \left| \sec \left(\frac{y}{x^2} \right) + \tan \left(\frac{y}{x^2} \right) \right| = -\frac{1}{x} + C$$

Ex:

the substitution $v = x^2 \ln y$ transforms the DE

$$x^2 y' + 4x^2 y \ln y = y \sin x$$

into...

$$v = x^2 \ln y \Rightarrow \frac{v}{x} = x \ln y$$

$$\frac{dv}{dx} = 2x \ln y + x^2 \frac{1}{y} \cdot \frac{dy}{dx}$$

$$y v' = 2xy \ln y + x^2 y^2 \Rightarrow y v' - 2xy \ln y = x^2 y^2$$

• put this in eq *

$$y v' - 2xy \ln y + 4x^2 y \ln y = y \sin x$$

$$v' - 2x \ln y + 4x^2 \ln y = \sin x$$

$$v' - 2 \frac{v}{x} + 4v = \sin x$$

$$v' + \left(4 - \frac{2}{x}\right) v = \sin x \quad \text{• (linear)}$$

Ex:

Solve the DE:

$$\frac{dx}{dy} - 2xy = e^{-3y^2} \cdot y (xe^{-y^2} + 3(xe^{-y^2})^2)$$

Use $v = xe^{-y^2}$

$$\rightarrow v = xe^{y^2} \rightarrow \frac{v}{e^{-y^2}} = x \rightarrow x = e^{y^2} \cdot v$$

$$\frac{dx}{dy} = 2ye^{y^2} v + e^{y^2} \frac{dv}{dy}$$

$$\rightarrow 2ye^{y^2} v + e^{y^2} \frac{dv}{dy} = 2y \cdot e^{y^2} v = e^{-3y^2} \cdot y (v + 3v^2)$$

$$e^{y^2} \frac{dv}{dy} = e^{-3y^2} y (v + 3v^2)$$

$$\frac{1}{v+3v^2} dv = y e^{-4y^2} dy \quad \text{separable}$$

Ch "3" :

linear DE of second order this DE has the form:

$$a(x)y'' + b(x)y' + c(x)y = g(x) \quad \text{--- (*)}$$

① If $g(x) = 0$, then the eq (*) becomes:

$$a(x)y'' + b(x)y' + c(x)y = 0$$

which is called the homog. linear DE of second order and if $g(x) \neq 0$. then it is called non homog.

Ex:

① $y'' + xy' + y = 0$ homog

② $y'' + 2xy - x = 0$

$y'' + 2xy = x$ non-homog

② If $a(x) = \overset{\text{const}}{a}$, $b(x) = \overset{\text{const}}{b}$, $c(x) = \overset{\text{const}}{c}$.

becomes

$ay'' + by' + cy = 0$, which is called homog linear of second order with constant coefficients.

Theorem: superposition principle (Homog)

If y_1, y_2 are solution for DE.

$$a(x)y'' + b(x)y' + c(x)y = 0$$

Then $y = C_1 y_1 + C_2 y_2$ is another solution.

⊛ The linear second order DE

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

can be written as $L[y](x) = g(x)$ where

$$L[y](x) = a(x)y'' + b(x)y' + c(x)y$$

is linear operator

Ex:

$$\text{If } L[y] = y'' + 2y' + 3y$$

$$\text{find } L[y](x) \text{ where } y(x) = x^3 - 2x$$

$$y = x^3 - 2x \rightarrow y' = 3x^2 - 2 \Rightarrow y'' = 6x$$

$$L[y](x) = L[x^3 - 2x] = 6x + 2(3x^2 - 2) + 3(x^3 - 2x)$$

$$= 6x + 6x^2 - 4 + 3x^3 - 6x$$

$$= 3x^3 + 6x^2 - 4$$

Theorem:

If $y_1, y_2, \dots, y_n$ are solutions for DE

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y = 0$$

Then $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$

is another solution.

Def: Wronskian

→ suppose that each of the functions $y_1, y_2, \dots, y_n$ possesses at least $n-1$ derivative, then the Wronskian of $y_1, y_2, \dots, y_n$ is given as:

$$W[y_1, y_2, \dots, y_n](x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Ex: find

$$\textcircled{1} W[e^x, x^2] = \begin{vmatrix} e^x & x^2 \\ e^x & 2x \end{vmatrix} = 2xe^x - x^2 e^x$$

Remark:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\textcircled{2} W[\sinh x, \cosh x] = \begin{vmatrix} \sinh x & \cosh x \\ \cosh x & \sinh x \end{vmatrix} = \sinh^2 x - \cosh^2 x = -1$$

$$\textcircled{3} W[x^3, x^2, e^x] = \begin{vmatrix} x^3 & x^2 & e^x \\ 3x^2 & 2x & e^x \\ 6x & 2 & e^x \end{vmatrix}$$

$$\Rightarrow x^3 \begin{vmatrix} 2x & e^x \\ 2 & e^x \end{vmatrix} - x^2 \begin{vmatrix} 3x^2 & e^x \\ 6x & e^x \end{vmatrix} + e^x \begin{vmatrix} 3x^2 & 2x \\ 6x & 2 \end{vmatrix}$$

$$+ e^x \begin{vmatrix} 3x^2 & 2x \\ 6x & 2 \end{vmatrix}$$

$$\Rightarrow x^3 (2xe^x - 2e^x) - x^2 (3x^2 e^x - 6xe^x) + e^x (6x^2 - 12x^2)$$



④ $W[x^3, x^2, e^x] =$

$$\begin{vmatrix} x^3 & x^2 & e^x \\ 3x^2 & 2x & e^x \\ 6x & 2 & e^x \end{vmatrix}$$

* طريقة ايزر
لحل المعادلات التفاضلية
رقم (3)

$$= \begin{vmatrix} x^3 & x^2 & e^x \\ 3x^2 & 2x & e^x \\ 6x & 2 & e^x \end{vmatrix} - \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \\ 6x & 2 \end{vmatrix}$$

$$= 2x^4 e^x + 6x^3 e^x + 6x^2 e^x - (12x^2 e^x + 2x^3 e^x + 3x^4 e^x)$$

$$= -x^4 e^x + 4x^3 e^x - 6x^2 e^x$$

$$= (-x^4 + 4x^3 - 6x^2) e^x$$

⑤ $W[3, x, x^2, x^3, x^4, x^5, e^{-x}]$

$$= \begin{vmatrix} 3 & x & x^2 & x^3 & x^4 & x^5 & e^{-x} \\ 0 & 1 & 2x & 3x^2 & 4x^3 & 5x^4 & -e^{-x} \\ 0 & 0 & 2 & 6x & 12x^2 & 20x^3 & e^{-x} \\ 0 & 0 & 0 & 6 & 24x & 60x^2 & -e^{-x} \\ 0 & 0 & 0 & 0 & 24 & 120x & e^{-x} \\ 0 & 0 & 0 & 0 & 0 & 120 & -e^{-x} \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-x} \end{vmatrix}$$

$$= 3 \cdot 1 \cdot 2 \cdot 6 \cdot 24 \cdot 120 e^{-x}$$

Def:

① A set of functions $y_1(x), \dots, y_n(x)$ is said to be linearly dependent on interval I , if there exist constants $c_1, c_2, \dots, c_n$ not all zero such that

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$

② If the set of functions is not linearly dependent, then it is said to be linearly independent. (if $c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$)
 $\Rightarrow$ all constant are equals zero.

Ex:

The functions: $y_1 = \cos^2 x$

$y_2 = \sec^2 x$, $y_3 = \tan^2 x$, $y_4 = \sin^2 x$

are linearly dependent on $(-\frac{\pi}{2}, \frac{\pi}{2})$

because:

there exist $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$ such that:

$$\rightarrow c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 = 0$$

$$c_1 \cos^2 x + c_2 \sec^2 x + c_3 \tan^2 x + c_4 \sin^2 x = 0$$

$$\cos^2 x - \sec^2 x + \tan^2 x + \sin^2 x = 0$$

$$\ast \sec^2 x - \tan^2 x = 1$$

$$\ast \sin^2 x + \cos^2 x = 1$$

Theorem:

we say that the set of functions

$y_1, y_2, \dots, y_n$ is linearly independent on interval I , if $W[y_1, y_2, \dots, y_n] \neq 0$ for any $x \in I$

Ex:

$I =$ the set of $y_1 = e^x, y_2 = e^{2x}$ linearly independent on $(-\infty, \infty)$?

yes, because

$$W[e^x, e^{2x}] = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$$= 2e^{3x} - e^{3x} = e^{3x} \neq 0$$

for all $x \in (-\infty, \infty)$

Ex: Is the set of $y_1 = e^{2x}, y_2 = 3e^{2x+1}$ linearly independent on $(-\infty, \infty)$?

NO, because

$$W[e^{2x}, 3e^{2x+1}] = \begin{vmatrix} e^{2x} & 3e^{2x+1} \\ 2e^{2x} & 6e^{2x+1} \end{vmatrix} = 6e^{4x+1} - 6e^{4x} = 0$$

for any $x \in (-\infty, \infty)$

Note: W def:

$$y_2 = 3e^{2x+1} = 3e \cdot e^{2x} = 3e \cdot y_1$$

∴ dependent.

Ex: The set of $y_1 = e^x$, $y_2 = 0$, $y_3 = \sin x$ is linearly dependent?

because:

$$W[e^x, 0, \sin x] = \begin{vmatrix} e^x & 0 & \sin x \\ e^x & 0 & \cos x \\ e^x & 0 & -\sin x \end{vmatrix}$$

$$= 0 \text{ for any } x$$

Remark:

$$\text{If } W[y_1, y_2, \dots, y_n] = 0$$

for any $x \in I \Rightarrow$ The set of $y_1, y_2, \dots, y_n$ is

linearly dependent.

Ex:

$\rightarrow$ the set $L_1(x) = x$, $L_2(x) = x-1$
and $L_3(x) = x+3$ linearly dependent
on $(-\infty, \infty)$?

Sol:

$$W[L_1, L_2, L_3] = W\left[\begin{matrix} x \\ x-1 \\ x+3 \end{matrix}\right]$$

$$= \begin{vmatrix} x & x-1 & x+3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{for any } x$$

$\Rightarrow$ dependent.

Def:

A set of functions $y_1, y_2, \dots, y_n$
is said to be fundamental set
of solution for DE.

Ex:

show that the set $\{x^{\frac{1}{2}}, x^{-1}\}$ is a fundamental
set of solution to DE:

$$2x^2 y'' + 3x y' - y = 0, \quad x > 0$$



* soln:

$$y_1 = \frac{1}{x^2} \rightarrow y_1' = \frac{1}{2} x^{-\frac{1}{2}} ; y_1'' = -\frac{1}{4} x^{-\frac{3}{2}}$$

put this in the left sides of DE

$$\begin{aligned} &= 2x^2 y'' + 3xy' - y = 2x^2 \left(-\frac{1}{4} x^{-\frac{3}{2}}\right) + 3x \left(\frac{1}{2} x^{-\frac{1}{2}}\right) - x^{\frac{1}{2}} \\ &= -\frac{1}{2} x^{\frac{1}{2}} + \frac{3}{2} x^{\frac{1}{2}} - x^{\frac{1}{2}} = 0 \quad \checkmark \end{aligned}$$

* $y_2 = x^{-1}$ is a solution.

$$* W[y_1, y_2] = W[x^{\frac{1}{2}}, x^{-1}]$$

$$\begin{aligned} &= \begin{vmatrix} x^{\frac{1}{2}} & x^{-1} \\ \frac{1}{2} x^{-\frac{1}{2}} & -x^{-2} \end{vmatrix} = -x^{-\frac{3}{2}} - \frac{1}{2} x^{-\frac{3}{2}} \\ &= -\frac{3}{2} x^{-\frac{3}{2}} \neq 0 \end{aligned}$$

 $\{x^{\frac{1}{2}}, x^{-1}\}$ is a fundamental set of solution.

Theorem (Abel's Theorem)

If y_1 and y_2 are solution of the DE

$$y'' + p_1(x)y' + q(x)y = 0$$

$$\text{then the } w[y_1, y_2](x) = Ce^{-\int p_1(x) dx}$$

Ex: If y_1, y_2 are solution for $y'' + \frac{2}{x}y' + y = 0$

Find the $w[y_1, y_2](x)$:

$$p_1(x) = \frac{2}{x}$$

$$w[y_1, y_2](x) = Ce^{-\int p_1(x) dx}$$

$$= Ce^{-\int \frac{2}{x} dx}$$

$$= Ce^{-2\ln x} = Cx^{-2} = \frac{C}{x^2}$$

Ex: If $\{y_1, y_2\}$ is independent set of solution for DE

$$xy'' + (\alpha - 1)y' + 3y = 0, \quad \alpha > 0$$

Find the $w[y_1, y_2](x)$ where $w[y_1, y_2](1) = e^{-1}$
 $\downarrow$
 $x \leftarrow w$

soln:

$$y'' + \frac{\alpha-1}{x}y' + \frac{3}{x}y = 0$$

$$P_1(x) = \frac{\alpha-1}{x} = 1 - \frac{1}{x}$$

$$w[y_1, y_2](x) = C e^{-\int P_1(x) dx}$$

$$= C e^{-\int (1 - \frac{1}{x}) dx}$$

$$= C e^{-x + \ln x} = C e^{-x} \cdot e^{\ln x}$$

$$= C \cdot x e^{-x}$$

Since $w[y_1, y_2](1) = e^{-1}$

$$C \cdot 1 \cdot e^{-1} = e^{-1}$$

$$\boxed{C=1}$$

$$w[y_1, y_2](x) = x e^{-x} \Rightarrow w[y_1, y_2](2) = 2e^{-2}$$

Reduction of order

If y_1 is a solution for DE

$$y'' + p_1(x)y' + q_1(x)y = 0$$

then the second solution is given by

$$y_2 = y_1 \cdot \int \frac{e^{-\int p_1(x) dx}}{y_1^2} dx$$

and the general solution is $y = c_1 y_1 + c_2 y_2$

Ex:

If $y_1 = x$ is a solution of DE: $x^2 y'' + 2xy' - 2y = 0$

$x > 0$, find the second solution and general solution.

$$y'' + y' \cdot \underbrace{\left(\frac{2x}{x^2}\right)}_{p_1(x)} - \frac{2}{x^2} y = 0 \quad \Rightarrow \quad p_1(x) = \frac{2x}{x^2} = \frac{2}{x}$$

$$\begin{aligned} y_2 &= y_1 \cdot \int \frac{e^{-\int p_1(x) dx}}{y_1^2} dx = x \cdot \int \frac{e^{-\int \frac{2}{x} dx}}{x^2} dx \\ &= x \cdot \int \frac{e^{-2 \ln x} = x^{-2}}{x^2} dx \end{aligned}$$



$$\begin{aligned} \leftarrow y_2 &= x \cdot \int x^{-1} dx \\ &= x \left(\frac{x^{-3}}{-3} \right) = -\frac{1}{3} x^{-2} \end{aligned}$$

The general solution is:

$$y = C_1 y_1 + C_2 y_2$$

$$= C_1 x + -\frac{1}{3} C_2 x^{-2} \checkmark = C_1 x + C_2 x^{-2} \checkmark$$

H.W

If $y_1 = e^x$ is a solution for DE
 $xy'' - (x+1)y' + y = 0$, $x > 0$
find the general solution.

linear, homogeneous second order DE with constant coefficients.

This DE has the form $a y'' + b y' + c y = 0$

suppose the solution has the form $y = e^{rx}$

$$\text{then } y' = r e^{rx} \quad , \quad y'' = r^2 e^{rx}$$

put this in above DE then we have .

$$a r^2 e^{rx} + b r e^{rx} + c e^{rx} = 0$$

$$(a r^2 + b r + c) e^{rx} = 0$$

$$\therefore a r^2 + b r + c = 0$$

is called the auxiliary eq of the DE

case 1: If $\Delta = b^2 - 4ac > 0$

then there is two different real roots

say r_1, r_2

The solutions are

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}$$

and the general solution (g.s) is

$$y = c_1 y_1 + c_2 y_2$$

Case 2: If $D = b^2 - 4ac = 0$ then there is one repeated real root:

Say, r_1, r_1 .

The solutions of DE are:

$$y_1 = e^{r_1 x}, \quad y_2 = x e^{r_1 x}$$

and the general solution is $y = c_1 y_1 + c_2 y_2$

Ex:

Find the general solution to the following DE

① $y'' - 4y' + 3y = 0$

$$y'' \rightarrow x^2$$

$$y' \rightarrow x$$

$$y \rightarrow 1$$

Solu 1

$$r^2 - 4r + 3 = 0$$

$$(r-1)(r-3) = 0 \quad r=1, r=3$$

$\Rightarrow$ the solution of D.E $\Rightarrow y_1 = e^x, y_2 = e^{3x}$

and

$$\begin{aligned} \text{the g.s is } y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^x + c_2 e^{3x} \end{aligned}$$

$$\textcircled{2} y'' - 9y = 0$$

$$r^2 - 9 = 0 \quad r = \pm 3$$

$$y_1 = e^{3x}, y_2 = e^{-3x}$$

$$\Rightarrow \text{g.s is } y = c_1 e^{3x} + c_2 e^{-3x}$$

$$\textcircled{3} y'' - 2y' = 0$$

$$r^2 - 2r = 0 \Rightarrow r(r-2) = 0 \quad r_1 = 0, r_2 = 2$$

$$\Rightarrow y_1 = e^{0x} = 1, y_2 = e^{2x} \quad \text{and}$$

$$\text{g.s is } y = c_1 + c_2 e^{2x}$$

$$\textcircled{4} y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0 \rightarrow (r-3)(r-3) = 0 \quad \boxed{r=3}$$

re

$$e^{3x}, (\cos \theta + i \sin \theta)$$

$\Rightarrow$ The solution are

$$y_1 = e^{3x}, \quad y_2 = x e^{3x}$$

and g.s is:

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

Case 3:

If $\Delta = b^2 - 4ac < 0$, then we have complex roots say $r = \alpha + i\beta$ where $i = \sqrt{-1}$

$\Rightarrow$ The solution are

$$y_1 = e^{\alpha x} \cos(\beta x), \quad y_2 = e^{\alpha x} \sin(\beta x)$$

and g.s is $= y = c_1 y_1 + c_2 y_2$

$$y = e^{ix}$$

$$y_1 = e^{ix} = e^{(\alpha + \beta i)x} = e^{\alpha x} \cdot e^{\beta x i}$$

$$= e^{\alpha x} (\cos(\beta x) + i \sin(\beta x))$$

$$y_2 = e^{(\alpha - \beta i)x} = e^{\alpha x} (\cos(\beta x) - i \sin(\beta x))$$

$$\frac{y_1 + y_2}{2} = e^{\alpha x} \cos(\beta x) \quad \frac{y_1 - y_2}{2i} = e^{\alpha x} \sin(\beta x)$$

Ex:

Find the g.s for

$$\textcircled{1} y'' - 4y' + 5y = 0$$

$$r^2 - 4r + 5 = 0$$

$$\Delta = b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot 5$$

$$= 16 - 20 = -4 < 0 \quad \text{complex roots}$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{(-4) \pm \sqrt{-4}}{2 \cdot 1} = \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

$$\alpha = 2, \beta = 1$$

$$\Rightarrow y = e^{ix} \cos \beta x = e^{2ix} \cos \alpha$$

$$y_1 = e^{2ix} \sin \alpha$$

$$\text{The g.s is: } y = c_1 y_1 + c_2 y_2$$

$$= e^{2ix} (c_1 \cos \alpha + c_2 \sin \alpha)$$

$$\textcircled{2} y'' - 2y' + 3y = 0$$

$$r^2 - 2r + 3 = 0$$

$$\Delta = b^2 - 4ac \Rightarrow (-2)^2 - 4 \times 1 \times 3 = 4 - 12$$
$$-8 < 0$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm \sqrt{-8}}{2 \times 1} = \frac{2 \pm 2\sqrt{2}i}{2}$$

$$r = 1 \pm \sqrt{2}i \quad \alpha = 1 \quad \beta = \sqrt{2}$$

$$\Rightarrow y_1 = e^x \cos(\sqrt{2}x)$$

$$y_2 = e^x \sin(\sqrt{2}x)$$

$$\text{g.s is } y = e^x (c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)$$

$$\textcircled{3} y'' + 9y = 0$$

$$r^2 + 9 = 0$$

$$r^2 = -9 \Rightarrow r = \pm \sqrt{-9} = \pm 3i \quad \alpha = 0$$

$$\beta = 3$$

$$\Rightarrow y_1 = \cos(3x) \quad y_2 = \sin(3x)$$

g.s is : $y = C_1 \cos 3x + C_2 \sin 3x$

Ex: r_1 r_2

If $e^{r_1 x}, e^{r_2 x}$ are solution.

For the DE $ay'' + by' + (c+4)y = 0$

find a, b, c

$$\Rightarrow r_1 = 3, r_2 = -2$$

$$(r - r_1)(r - r_2) = 0$$

$$(r - 3)(r + 2) = 0$$

$$r^2 + 2r - 3r - 6 = 0$$

$$r^2 - r - 6 = 0$$

$$y'' - y' - 6y = 0 \Rightarrow ay'' + by' + (c+4)y = 0$$

$$\boxed{a=1}, \boxed{b=-1}, c+4=-6$$

$$\boxed{c=-10}$$

Ex.

If $x e^{2x}$ is a solution for DE

$$y'' + (b-1)y' + 2cy = 0$$

Find the value of b, c .

$$\therefore r_1 = 2, r_2 = 2$$

$$(r-r_1)(r-r_2) = 0$$

$$(r-2)(r-2) = 0 \rightarrow r^2 - 4r + 4 = 0$$

$$y'' - 4y' + 4y = 0$$

$$\begin{cases} b-1 = -4 \\ b = -3 \end{cases}$$

$$\begin{cases} 2c = 4 \\ c = 2 \end{cases}$$

Ex.

If $y = e^{-2x} \cos(3x)$ is a solution for DE

$$y'' + by' + cy = 0 \text{ Find the eq}$$

complex roots:

$$r = \alpha \pm \beta i$$

$$r = -2 \pm 3i$$

$$\begin{cases} i = \sqrt{-1} \\ i^2 = -1 \end{cases}$$

$$r+2 = \pm 3i$$

$$(r+2)^2 = (\pm 3i)^2$$

$$r^2 + 4r + 4 = 9(-1)$$

$$r^2 + 4r + 13 = 0$$

$$y'' + 4y' + 13y = 0$$

Ex: solve

the following I.V.P $y'' + 3y' + 2y = 0$,

where $y(0) = 3$, $y'(0) = -4$

$$r^2 + 3r + 2 = 0 \rightarrow (r+1)(r+2) = 0$$

$$r_1 = -1, \quad r_2 = -2$$

$$\Rightarrow y_1 = e^{-x}, \quad y_2 = e^{-2x}$$

$$\text{g.s. is: } y = c_1 e^{-x} + c_2 e^{-2x}.$$

$$\text{Since } y(0) = 3 \Rightarrow 3 = c_1 + c_2$$

$$\text{where } y'(0) = -4 \Rightarrow y' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

$$-4 = -c_1 - 2c_2$$

$$3 = c_1 + c_2$$

$$-4 = -c_1 - 2c_2$$

$$-1 = -c_2$$

$$\boxed{c_2 = 1}$$

$$\text{put this in } 3 = c_1 + c_2$$

$$3 = c_1 + 1$$

$$\boxed{c_1 = 2}$$

$$y = 2e^{-x} + e^{-2x}$$

the solution
of I.V.P

H.W:

solve the I.P.V.

$$y'' + 9y = 0, \quad y\left(\frac{\pi}{3}\right) = 2 \quad \text{and} \quad y'\left(\frac{\pi}{3}\right) = -4$$

Euler Eq. (homog, linear, second order)

This DE has the form:

$$ax^2y'' + bxy' + cy = 0, \quad x > 0$$

The substitution $| x = e^t \Leftrightarrow \ln x = t |$

transforms the Euler eq. into second order linear DE with constant coefficients

of the form:

$$a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$$

Now Note that,

$$| y \rightarrow t \rightarrow x |$$

$$\bullet \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$\bullet \frac{d^2 y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d}{dx} \left[\frac{dy}{dt} \cdot \frac{1}{x} \right] = \frac{dy}{dt} \cdot \frac{dt}{dx} \cdot \frac{1}{x} + \frac{dy}{dt} \cdot \frac{1}{x^2}$$

$$\frac{dy'}{dx^2} = \frac{1}{x^2} \frac{d^2y}{dt^2} - \frac{1}{x^2} \frac{dy}{dt}$$

$$\rightarrow x^2 \frac{dy'}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

put this in DE:

$$ax^2 y'' + bxy' + cy = 0$$

$$a \left(\frac{d^2y}{dt^2} - \frac{dy}{dt} \right) + b \frac{dy}{dt} + cy = 0$$

$$a \frac{d^2y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$$

Ex:

find the g.s for.

$$(1) x^2 y'' - 3xy' - 5y = 0$$

$$xy' \rightarrow r$$

$$x^2 y'' \rightarrow (r-1)$$

$$x^3 y''' \rightarrow r(r-1)(r-2)$$

$$\text{Euler eq} \Rightarrow ax^2 \leftrightarrow (ax)^2$$

$$\Downarrow$$

$$a \frac{dy^2}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$$

$$\frac{d^2y}{dt^2} - 4 \frac{dy}{dt} + 5y = 0$$

$$r^2 - 4r - 5 = 0, (r-5)(r+1) = 0 \Rightarrow r_1 = 5$$

$$r_2 = -1$$



$$y_1(t) = e^{5t}, \quad y_2(t) = e^{-t}$$

$$y_1(x) = x^5, \quad y_2(x) = x^{-1}$$

$$y = C_1 x^5 + C_2 x^{-1}$$

$$\rightarrow \textcircled{2} \quad x^2 y'' + 5xy' + 4y = 0, \quad x > 0$$

$$x \frac{dy^2}{dx^2} + (b-a) \frac{dy}{dx} + Cy = 0$$

$$\frac{dy^2}{dx^2} + 4 \frac{dy}{dx} + 4y = 0$$

$$r^2 + 4r + 4 = 0 \Rightarrow (r+2)(r+2) = 0 \quad \boxed{r = -2}$$

$$y_1(t) = e^{-2t}, \quad y_2(t) = t e^{-2t}$$

$$y_1(x) = x^{-2}, \quad y_2(x) = (\ln x) x^{-2} \quad x^{-2} \ln x$$

$$\text{g.s is } y = C_1 x^{-2} + C_2 x^{-2} \ln x$$

$$3) 2x^2 y'' + 2xy' + 8y = 0, \quad x > 0$$

Enter eq $\rightarrow x = e^t \Leftrightarrow \ln x = t$

$$2 \frac{dy^2}{dt^2} + 8y = 0$$

$$2r^2 + 8 = 0 \rightarrow r^2 = -4$$

$$r = \pm \sqrt{-4} = \pm 2i$$

$$\alpha = 0, \beta = 2$$

$$y, b = e^{\alpha t} \cos(\beta t)$$

$$y_1(t) = 2 \cos(2t)$$

$$y_2(t) = 2 \sin(2t)$$

$$y_2(t) = 2 \sin(2 \ln x)$$

$$\rightarrow y_1, y_2 \text{ is}$$

$$y = 2 \cos(2 \ln x) + 2 \sin(2 \ln x)$$

Ex: If x^2, x^3 are solutions for

$$ax^2 y'' + bxy' + cy = 0 \quad \text{Find } a, b, c.$$

Enter eq $\rightarrow x = e^t \Leftrightarrow \ln x = t$

Since $y_1(x) = x^2 \ln x$ is a solution then

$y_1(t) = t e^{-2t}$ is a solution. for

$$a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0 \quad r^2 - 2r = 0$$

$$(r-2)(r+2) = 0 \rightarrow r^2 - 4r + 4 = 0$$

$a = 1$
$b = 5$
$c = 4$

Ex:

If $y = x^2 \cos(\ln x)$ is a solution for

$$ax^2 y'' + bxy' + cy = 0$$

find this DE:

$$\text{Euler eq} \Rightarrow \boxed{x = e^t \Leftrightarrow t = \ln x}$$

Since $y(x) = x^2 \cos(\ln x)$ is a solution.

$$\Rightarrow y(t) = e^{2t} \cos t$$

is a solution for DE

$$a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$$

$$\rightarrow \text{Complex roots} \Rightarrow r = \alpha \pm \beta i \rightarrow r = 2 \pm i$$

$$r = 2 \pm i \rightarrow r - 2 = \pm i$$

$$(r-2)^2 = (\pm i)^2$$

$$r^2 - 4r + 4 = -1 \rightarrow r^2 - 4r + 5 = 0$$

$$\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 5y = 0$$

$$a = 1, \quad b = -3, \quad c = 5$$

the DE is

$$\Rightarrow x^2 y'' - 3xy' + 5y = 0$$

Ex:

If x^{-2}, x^3 are solutions for

$ax^2y'' + bxy' + cy = 0$ find a, b, c .

Enter by $\boxed{x = e^t \Leftrightarrow \ln x = t}$

$$\Rightarrow y_1(x) = x^{-2}, \quad y_2(x) = x^3$$

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{3t}$$

are solution for DE.

$$a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$$

$$r_1 = -2, \quad r_2 = 3 \Rightarrow (r+2)(r-3) = 0$$

$$(r+2)(r-3) = 0$$

$$r^2 - r - 6 = 0$$

$$\frac{d^2 y}{dt^2} - \frac{dy}{dt} - 6y = 0$$

$$a = 1, \quad b = 0, \quad c = -6$$

Ex: If $w[x, y(x)] = x$. find $y(x)$
where $y(1) = 5$ $\downarrow$
 $x > 0$

$$w[x, y(x)] = x \quad \left| \begin{array}{cc} x & y \\ 1 & y' \end{array} \right| = x$$

$$x y' - y = x$$

$$y' - \frac{1}{x} y = 1$$

linear first order DE

$$p(x) = -\frac{1}{x} \quad Q(x) = 1$$

$$\therefore M(x) = e^{\int p(x) dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$y = \frac{1}{x} \left(\int \int M(x) Q(x) dx + C \right)$$

$$= \frac{1}{x} \left(\int \frac{1}{x} dx + C \right)$$

$$\therefore y = x (\ln x + C) \Rightarrow y = x \ln x + Cx$$

$$\text{since } y(1) = 5 \Rightarrow 5 = 1 \cdot \ln 1 + C(1) \quad \boxed{C = 5}$$

$$y(x) = x \ln x + 5x \quad \left| \underline{\underline{y(2) = 2 \ln 2 + 10}} \right|$$

Ex:

If $W[2, x, f(x)] = x^3$ and $f(0) = 1$

$f'(0) = 3$ find $f(x)$

$$W[2, x, f(x)] = x^3 \quad \left| \begin{array}{cc|c} 2 & x & f(x) \\ 0 & 1 & f'(x) \\ 0 & 0 & f''(x) \end{array} \right| = x^3$$

$$\therefore 2f''(x) = x^3$$

$$f''(x) = \frac{x^3}{2}$$

$$\therefore f'(x) = \int \frac{1}{2} x^3 dx \Rightarrow f'(x) = \frac{1}{8} x^4 + C$$

$$\text{Since } f'(0) = 3 \Rightarrow f'(0) = \frac{1}{8} (0)^4 + C \quad \boxed{C=3}$$

$$\therefore f''(x) = \frac{1}{8} x^4 + 3 \Rightarrow f(x) = \int \left(\frac{1}{8} x^4 + 3 \right) \cdot dx$$

$$f(x) = \frac{1}{40} x^5 + 3x + C_1$$

$$\text{Since } f(0) = 1 \Rightarrow f(0) = \frac{1}{40} (0)^5 + 3(0) + C_1 = 1 \quad \boxed{C_1=1}$$

$$f(x) = \frac{1}{40} x^5 + 3x + 1$$

Non homogeneous - linear DE of second order.

To solve the non-homog DE:

$$ay'' + by' + cy = g(x)$$

we use two methods:

- ① undetermined coefficients
- ② variation of parameters

Method ①: undetermined coefficients

$$ay'' + by' + cy = g(x)$$

- ① solve the homog eq $ay'' + by' + cy = 0$
to find the homog solution say,

$$y_h = C_1 \cdot y_1 + C_2 y_2$$

- ② solve the non-homog DE to find the particular solution say y_p

- ③ The general solution is:

$$y = y_h + y_p$$

Remark:

In this method, $g(x)$ must be exponential, polynomial, or $\sin x, \cos x$.

To find y_p :

$g(x)$

y_p

polynomial

2



$(A)x^k$

$x+2$



$(Ax+B)x^k$

x^2+2x



$(Ax^2+Bx+C)x^k$

k: عدد صحيح موجب (العدد) في جذر المعادلة التفاضلية.

$g(x)$

y_p

$2e^{ax}$



$(A(e^{ax}))x^m$

$(x)e^{ax}$



$((Ax+B)e^{ax})x^m$

$(x^2+x)e^{ax}$



$((Ax^2+Bx+C)e^{ax})x^m$

m: عدد صحيح موجب (العدد) في جذر المعادلة التفاضلية.

g(x)

y
p

$\sin(Bx)$ or $\cos(Bx)$

$$\sin(Bx) \xrightarrow{\alpha^h} (A \cos(Bx) + B \sin(Bx))$$

$$x \sin(Bx) \xrightarrow{\alpha^h} ((A\alpha + B) \cos Bx + (C\alpha + D) \sin(Bx))$$

$$2e^{\alpha x} \sin(Bx) \xrightarrow{\alpha^h} (e^{\alpha x} (A' \cos Bx + D' \sin(Bx)))$$

$$\alpha e^{\alpha x} \sin(Bx) \xrightarrow{\alpha^h} (e^{\alpha x} ((A\alpha + B) \cos Bx + (C\alpha + D) \sin Bx))$$

h : عدد صحیح (α + Bi) کے لیے، جہاں α و B حقیقی اور B ≠ 0

Ex:

solve the following DE:

$$y'' + 4y' = 2x^2 - 3x + 5e^{-4x}$$

• homog

$$y'' + 4y' = 0$$

$$r^2 + 4r = 0$$

$$r(r+4) = 0$$

$$r_1 = 0, r_2 = -4$$

$$y_1 = 1, y_2 = e^{-4x}$$

$$y_h = c_1 + c_2 e^{-4x}$$

• non homog

$$g(x) = 2x^2 - 3x + 5e^{-4x}$$

$$y_p = x(Ax^2 + Bx + C) + (De^{-4x})x$$

$$y_p = Ax^3 + Bx^2 + Cx + Dx e^{-4x}$$

since y_p is a solution, a form of particular

$$y_p' = 3Ax^2 + 2Bx + C + De^{-4x} - 4Dxe^{-4x}$$

$$y_p'' = 6Ax + 2B - 4De^{-4x} - 4Dxe^{-4x} + 16Dxe^{-4x}$$

$$y''_p = 6Ax + 2B - 8De^{-4x} + 16Dxe^{-4x}$$

Now, put this in DE:

$$\Rightarrow 6Ax + 2B - 8De^{-4x} + 16Dxe^{-4x} + 4(3Ax^2 + 2Bx + C + De^{-4x} - 4Dxe^{-4x})$$

$$= 2x^2 - 3x + 5e^{-4x}$$

$$\Rightarrow 6Ax + 2B - 8De^{-4x} + 16Dxe^{-4x} + 12Ax^2 + 8Bx + 4C + 4De^{-4x} - 16Dxe^{-4x} = 2x^2 - 3x + 5e^{-4x}$$

$$\Rightarrow 6Ax + 2B - 4De^{-4x} + 12Ax^2 + 8Bx + 4C$$

$$= 2x^2 - 3x + 5e^{-4x}$$

$$6A + 8B = -3 \Rightarrow 6\left(\frac{1}{6}\right) + 8B = -5$$

$$\boxed{B = -\frac{1}{2}}$$

$$2B + 4C = 0, \quad 12A = 2$$

$$\boxed{A = \frac{1}{6}}$$

$$-4D = 5$$

$$\boxed{D = -\frac{5}{4}}$$

$$2\left(-\frac{1}{2}\right) + 4C = 0$$

$$\boxed{C = \frac{1}{4}}$$

Ex:

Find the general form of particular solution for:

$$\textcircled{1} y'' - 3y' - 4y = e^{4x} - 3x$$

homog

$$\rightarrow r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0 \rightarrow (r=4)(r=-1)$$

$$y_1 = e^{4x}, y_2 = e^{-x}$$

non homog:

$$\rightarrow g(x) = e^{4x} - 3x \rightarrow e^{4x} - e^{x(\ln 3)}$$

$$y_p = (Ae^{4x})x + Be^{x(\ln 3)}$$

$$\begin{aligned} 3x &= e^{\ln(3)x} \\ &= e^{x(\ln 3)} \end{aligned}$$

$$\textcircled{2} x^3 y'' - 2x^2 y' + 2xy = x^3 - x^2 \ln x + 5$$

$$x^2 y'' - 2x y' + 2y = x^2 \ln x + 5x^{-1}$$

$$\text{Euler eq} \Rightarrow \boxed{x = e^b \Leftrightarrow \ln x = b}$$

$$a=1$$

$$b=2$$

$$c=2$$

$$\frac{a dy^2}{db^2} + (b-a) \frac{dy}{db} + cy = g(b)$$

$$\frac{dy^2}{db^2} - 3 \frac{dy}{db} + 2y = e^{2b} \cdot b^2 + 5e^{-b}$$



homog: $r^2 - 3r + 2 = 0$

$$(r-2)(r-1) = 0$$

$$r_1 = 2, \quad r_2 = 1$$

$$y_1(t) = e^{2t}, \quad y_2(t) = e^t$$

$$y_1(x) = x^2$$

$$y_2(x) = x$$

non homog:

$$g(t) = e^{2t} - t e^t + 5e^{-t}$$

$$y_p(t) = (A e^{2t}) t + ((Bt + C) e^t) t + D e^{-t}$$

$$y_{p,x} = A x^2 \ln x + B (\ln x)^2 + C \ln x + D x^{-1}$$

③ $y'' + 4y = \sin x \cos x$

$$y'' + 4y = \frac{1}{2} \sin 2x$$

homog:

$$r^2 + 4 = 0$$

$$r = \pm \sqrt{-4} = \pm 2i \quad \alpha = 0, \quad \beta = 2$$

$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

non homog: $y(x) = \frac{1}{2} \sin 2x$

$$y_p = A \cos 2x + D \sin 2x$$

$$y_p = A x \cos 2x + D x \sin 2x$$

H.W:

$$\begin{aligned} 4. \quad y'' - 4y &= \sin 2x \cdot \cos 3x \\ &= \frac{1}{2} [\sin(-x) + \sin(5x)] \end{aligned}$$

$$5. \quad y'' + 36y = \sin^2(3x) = \frac{1}{2} - \frac{1}{2} \cos 6x$$

$$6. \quad y'' - 2y' + y = e^x$$

$$7. \quad y'' - 4y = \cosh 2x = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

Ex:

Find the values of a, b, c and d so that $y(x) = e^{-x} + 2e^{2x} + \sin(4x)$ is a solution to the DE.

$$y'' + ay' + by = c \sin(4x) + d \cos(4x)$$

$$y_h = e^{-x} + 2e^{2x}$$

$$\rightarrow y_1 = e^{-x}, \quad y_2 = e^{2x}$$

$$\therefore r_1 = -1 \quad , \quad r_2 = 2$$

$$(r - r_1)(r - r_2) = 0 \rightarrow (r + 1)(r - 2) = 0$$

$$r^2 - r - 2 = 0 \rightarrow y'' - y' - 2y = 0$$

$$y'' + ay' + by = 0$$

$$\boxed{a = -1, \quad b = -2}$$

$$* y_p = \sin 4x$$

is solution for nonhomog DE:

$$y'' + y' - 2y = c \sin 4x + d \cos 4x,$$

$$\therefore y_p' = 4 \cos 4x \quad \text{and} \quad y_p'' = -16 \sin 4x,$$

$$-16 \sin 4x - 4 \cos 4x - 2 \sin 4x = c \sin 4x + d \cos 4x$$

$$\boxed{c = -18} \quad \boxed{d = -4}$$

معامل $\sin$

معامل $\cos$

Variation of parameters

$$ay'' + by' + cy = f(x) \Rightarrow g(x) = \frac{f(x)}{a}$$

① Find the homogy solutions

$$y_1, y_2 \text{ and } y_h = c_1 y_1 + c_2 y_2$$

$$\textcircled{2} \text{ Find } W[y_1, y_2] \propto \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$\textcircled{3} v_1 = - \int \frac{y_2 \cdot g(x)}{W} dx$$

$$v_2 = \int \frac{y_1 \cdot g(x)}{W} dx$$

④ the particular solution is

$$y_p = y_1 v_1 + y_2 v_2$$

⑤ the g.s is: $y = y_h + y_p$

Ex:

Find the general solution to DE:

$$y'' + 9y = \sec(3x)$$

① homog:

$$r^2 + 9 = 0, \quad r = \pm \sqrt{-9} = \pm 3i$$

$$y_1 = e^{3ix} \cos(3x) = \cos(3x)$$

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

$$\textcircled{2} W[\cos(3x), \sin(3x)] = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2(3x) + 3\sin^2(3x)$$

$$\Rightarrow 3 \cdot 1 = 3$$

$$\textcircled{3} g(x) = \sec(3x) = \sec(3x)$$

$$v_1 = - \int \frac{y_2 \cdot g(x)}{W} dx = - \int \sin 3x \cdot \sec(3x) dx$$

$$= \frac{-1}{-3 \cdot 3} \int \frac{-3 \sin 3x}{\cos 3x} dx = \frac{1}{9} \ln |\cos(3x)|$$



$$v_2 = \int \frac{y_1 g(x)}{w} dx$$

$$= \int \frac{\cos(3x) \cdot \sec(3x)}{3} dx$$

$$= \int \frac{1}{3} dx = \frac{1}{3} x$$

• particular solution is

$$y_p = y_1 v_1 + y_2 v_2$$

$$= \frac{1}{9} \cos(3x) - \ln|\cos(3x)| + \frac{1}{3} x \sin(3x)$$

⑤ The g.s is;

$$y = y_h + y_p$$

Ex:

solve the DE $y'' - 2y' + y = \frac{e^x}{x^2 + 1}$

① homog

$$r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0$$

$$r_1 = 1, r_2 = 1$$

$$\Rightarrow y_1 = e^x, y_2 = x e^x$$

$$y_h = c_1 e^x + c_2 x e^x$$

② $w[y_1, y_2] = w[e^x, x e^x] = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix}$

$$= e^{2x} + x e^{2x} - x e^{2x} = e^{2x}$$

③ $g(x) = \frac{e^x}{x^2 + 1}$

$$\Rightarrow v_1 = - \int \frac{y_2 \cdot g(x)}{w} dx = - \int \frac{x e^x \cdot \frac{e^x}{x^2 + 1}}{e^{2x}} dx$$
$$= - \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = - \frac{1}{2} \ln |x^2 + 1|$$

$$v_2 = \int \frac{y_1 \cdot g(x)}{w} dx = \int \frac{e^x \cdot \frac{e^x}{x^2 + 1}}{e^{2x}} dx$$

$$= \int \frac{1}{x^2 + 1} dx = \tan^{-1} x$$

The particular solution is:

$$y_p = y_1 v_1 + y_2 v_2$$

$$= -\frac{1}{2} e^x \ln(x^2 + 1) + x e^x \tan^{-1} x$$

The g.s is $y = y_h + y_p$

Ex:

Solve the DE

$$x^2 y'' - 2xy' - 4y = x^3 - 2x, \quad x > 0.$$

"H.W"

Ex: Find the Euler equation second order
 having the general solution $y_{gs} = \underbrace{c_1 x + c_2 x \ln x}_{y_h} + \underbrace{x(1-\ln x)^2}_{y_p}$

Euler $x = e^t \leftrightarrow \ln x = t$ y_p

$$y_{gs} = c_1 x + c_2 x \ln x + x(1-\ln x)^2$$

$$y(t) = \underbrace{c_1 e^t + c_2 e^t \cdot t}_{y_h} + e^t \cdot t^2$$

Since $y_h = c_1 e^t + c_2 t e^t$ y_h

$$r_1 = 1, r_2 = 1 \quad \text{r.s.}$$

$$(r-r_1)(r-r_2) = 0$$

$$(r-1)(r-1) = 0 \rightarrow r^2 - 2r + 1 = 0$$

$$\frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + y = 0$$

$$\frac{a dy''}{dt^2} + (b-a) \frac{dy}{dt} + cy = 0$$

$$a = 1, b-a = -2, c = 1$$

$$b-1 = -2$$

$$b = -1$$

$$ax^2 y'' + bxy' + cy = 0$$

$$x^2 y'' - x y' + y = 0$$

Since $y_p(t) = t^2 e^t$

is a particular solution of eq:

$$\frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + y = g(t)$$

$$* y_p' = 2t e^t + t^2 e^t$$

$$y_p'' = 2e^t + 2t e^t + 2t e^t + t^2 e^t$$

$$= 2e^t + 4t e^t + t^2 e^t$$

$$\rightarrow 2e^t + \cancel{4t e^t} + \cancel{t^2 e^t} - \cancel{4t e^t} - \cancel{2t^2 e^t} + \cancel{t^3 e^t} = g(t)$$

$$\rightarrow g(t) = 2e^t$$

$$g(x) = 2x$$

$$\therefore x^2 y'' - x y' + y = 2x$$

higher order DE (homog - non homog).

Ex:

Find the general solution to the following DE

$$\textcircled{1} y''' + 3y'' + 3y' + y = 0$$

$$r^3 + 3r^2 + 3r + 1 = 0 \Rightarrow \underline{r^3 + 1} + \underline{3r^2 + 3r} = 0$$

$$\rightarrow (r+1)(r^2 - r + 1) + 3r(r+1) = 0$$

$$(r+1)(r^2 - r + 1 + 3r) = 0$$

$$(r+1)(r^2 + 2r + 1) = 0$$

$$(r+1)(r+1)(r+1) = 0 \rightarrow \underset{\substack{\uparrow \\ \text{So}}}{r_1 = -1}, \underset{\substack{\uparrow \\ \text{So}}}{r_2 = -1}, \underset{\substack{\uparrow \\ \text{So}}}{r_3 = -1}$$

$$y_1 = e^{-x}, y_2 = x e^{-x}, y_3 = x^2 e^{-x}$$

$$\text{g.s} = y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

$$\textcircled{2} y^{(3)} - 6y'' + 11y' - 6y = 0$$

$$r^3 - 6r^2 + 11r - 6 = 0 \quad (r=1) \text{ because}$$

$$(1)^3 - 6(1)^2 + 11(1) - 6 = 0$$

$$r^2 - 5r + 6 = 0 \leftarrow$$

$$(r-3)(r-2) = 0$$

$$r_1 = 3, r_2 = 2, r_3 = 1$$

$$\begin{array}{cccc} 1 & -6 & 11 & -6 \\ & \downarrow & & \\ & 1 & -5 & 6 \\ & & & \boxed{0} \end{array}$$

$$y_1 = e^x, y_2 = e^{3x}, y_3 = e^{2x}$$

$$\text{g.s is } y = c_1 e^x + c_2 e^{3x} + c_3 e^{2x}$$

$$\textcircled{3} y^{(3)} + 3y'' - 4y = 0$$

$$r^3 + 3r^2 - 4 = 0 \quad (r=1) \text{ because } (1)^3 + 3(1)^2 - 4 = 0$$

$$\begin{array}{cccc} 1 & 3 & 0 & -4 \\ & \downarrow & & \\ & 1 & 4 & 4 \\ & & & \boxed{0} \end{array}$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0$$

$$r_1 = -2, r_2 = -2$$

$\Rightarrow$

$$r_1 = 1 \rightarrow r_2 = -2, r_3 = -2$$

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$$

$$(4) y''' + y = 0$$

$$r^3 - 1 = 0$$

$$(r-1)(r^2+r+1) = 0 \rightarrow \boxed{r=1}$$

↓

$$\Delta = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0 \text{ complex}$$

$$r = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$$

α β

$$y_1 = e^x, y_2 = e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right), y_3 = e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$y = c_1 e^x + c_2 \cdot e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \cdot e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$(5) \quad y^{(4)} + 2y^{(2)} + y = 0$$

$$r^4 + 2r^2 + 1 = 0$$

$$(r^2 + 1)(r^2 + 1) = 0$$

↓

↓

$$r^2 + 1 = 0$$

$$r^2 + 1 = 0$$

$$r = \pm \sqrt{-1}$$

$$r = \pm i$$

$$r = \pm i$$

$$\alpha = 0, \beta = 1$$

$$\alpha = 0, \beta = 1$$

$$y_3 = \alpha \cos \alpha$$

$$y_1 = \cos \alpha$$

$$y_4 = \alpha \sin \alpha$$

$$y_2 = \sin \alpha$$

$$y = C_1 \cos \alpha + C_2 \sin \alpha + C_3 \alpha \cos \alpha + C_4 \alpha \sin \alpha$$

Higher order non homog.

Ex:

Find the general form of particular solution for

$$\textcircled{1} y''' + -2y'' + y' = x + ze^{2x} - \cos 3x$$

homog:

$$r^3 - 2r^2 + r = 0 \rightarrow r(r^2 - 2r + 1) = 0$$

$$r(r-1)(r-1) = 0$$

$$r_1 = 0, r_2 = 1, r_3 = 1$$

$$y_1 = 1, y_2 = ze^{2x}, y_3 = xe^{2x}$$

non homog:

$$g(x) = x + ze^{2x} - \cos(3x)$$

$$\begin{aligned} g_p &= (Ax + B)x + De^{2x} + H \cos(3x) + J \sin(3x) \\ &= Ax^2 + Bx + De^{2x} + H \cos 3x + J \sin 3x \end{aligned}$$

$$\textcircled{2} y^{(4)} + y^{(3)} = z + e^x$$

$$\text{homog: } r^4 + r^3 = 0$$

$$r^3(r+1) = 0 \quad r_1 = 0, r_2 = 0, r_3 = 0, r_4 = -1$$

$$y_1 = 1, y_2 = x, y_3 = x^2, y_4 = e^{-x}$$

non homog :

$$\rightarrow g(x) = 2x e^{-x}$$

$$y_p = (A)x^3 + (De^{-x})x$$

$$y_p = Ax^3 + Dx e^{-x}$$

$$3) y^{(4)} - 2y^{(3)} + y^{(2)} = 34e^x$$

homog :-

$$r^4 - 2r^3 + r^2 = 0 \rightarrow r^2(r^2 - 2r + 1) = 0$$

$$r_1 = 0, r_2 = 0, r_3 = 1, r_4 = 1$$

$$y_1 = 1, y_2 = x, y_3 = e^x, y_4 = x e^x$$

non homog:

$$g(x) = 34e^x$$

$$y_p = Ax^2 + (De^x)x$$

$$y_p = Ax^2 + Dx^2 e^x$$

$$\textcircled{4} \quad y^3 + 6y'' + 9y' + 4y = e^{-x}$$

homog =

$$r^3 + 6r^2 + 9r + 4 = 0$$

$$r_1 = -1 \quad \text{because } (-1)^3 + 6(-1)^2 + 9(-1) + 4 = 0$$

$$\begin{array}{cccc} \boxed{-1} & 1 & 6 & 9 & 4 \\ & \downarrow & -1 & -5 & -4 \\ & & 1 & 5 & 4 & \boxed{10} \end{array}$$

$$* \quad r^2 + 5r + 4 = 0$$

$$(r+1)(r+4) = 0$$

$$r_2 = -1, \quad r_3 = -4$$

$$y_1 = e^{-x}, \quad y_2 = x e^{-x}, \quad y_3 = e^{-4x}$$

non homog:

$$g(x) = e^{-x}$$

$$y_p = (A e^{-x}) x^2 \rightarrow A x^2 e^{-x}$$

Ex: IF $(r+7)(r^2+1)^2(r^2+4) = 0$ is the auxiliary equation for the DE $[y]^{(5)} = 0$ with constant coefficients

Find the order of this DE? $\textcircled{7}$

② Find the y_h solution?

$$(r+7)(r^2-1)^2(r^2+4)=0$$

$$r_1 = -7$$

↓

$$(r^2-1)^2=0$$

$$(r^2-1)(r^2-1)=0$$

$$r^2=1 \quad r^2=1$$

$$r=\pm 1 \quad r=\pm 1$$

$$r_2=1 \quad r_4=1$$

$$r_3=-1 \quad r_5=-1$$

$$r^2+4=0$$

$$r^2=-4$$

$$r=\pm 2i$$

$$\alpha=0, \beta=2$$

$$y_1 = e^{-7x}, y_2 = e^x, y_3 = e^{-x}, y_4 = x e^x, y_5 = x e^{-x}$$

$$y_6 = \cos 2x, y_7 = \sin 2x$$

$$y_h = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 + c_5 y_5 + c_6 y_6 + c_7 y_7$$

③ Find the form of particular solution

$$\text{for } I[y](x) = e^x + 3x e^{-x} + \sin 2x$$

$$g(x) = e^x + 3x e^{-x} + \sin 2x$$

$$y_p = (A e^x) x^2 + ((Bx+C) e^{-x}) x^2 + K (D \cos 2x + H \sin 2x)$$

Ex:

Find the homogeneous DE with constant coefficients of at least order with has the solutions

$$y = 7x^2 + 9x e^{3x} - x \sin 2x$$

↓

$$r_1 = 0$$

$$r_2 = 0$$

$$r_3 = 0$$

$$\boxed{r^3 = 0}$$

↓

$$r_4 = 3$$

$$r_5 = 3$$

$$\boxed{(r-3)^2 = 0}$$

↘

Complex

$$\alpha = 0, \beta = 2$$

$$r = \alpha \pm \beta i$$

$$r = \pm 2i \rightarrow r = \pm 2i$$

⇒

$$r^2 = (\pm 2i)^2$$

$$r^2 = -4$$

$$r^2 + 4 = 0$$

$$\Rightarrow r^3 (r-3)^2 + (r^2 + 4) = 0$$

$$y^{(9)} = 6y^{(8)} + 12y^{(7)} + 48y^{(6)} + 88y^{(5)} + -96y^{(4)}$$

$$+ 144y^{(3)} = 0$$

Ch. 4

Laplace Transform

Def:

Let $f(t)$ be a function on $[0, \infty)$, then
Laplace transform of $f(t)$, written
as $\mathcal{L}\{f(t)\}$ or $F(s)$, is defined by

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt = F(s)$$

Remark:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

improper integral

$$= \lim_{h \rightarrow \infty} \int_0^h f(t) e^{-st} dt$$

Ex:

Find Laplace transform for $f(t) = t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\}$$

$$= \int_0^{\infty} t e^{-st} dt$$

$$= \lim_{h \rightarrow \infty} \int_0^h t e^{-st} dt$$

1	$\int$
$\oplus t$	e^{-st}
$\ominus 1$	e^{-st}
$\ominus$	$\frac{e^{-st}}{-s}$
	$\frac{e^{-st}}{s^2}$

$$= \lim_{h \rightarrow \infty} \left[-\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^h$$

$$= \lim_{h \rightarrow \infty} \left[-\frac{h e^{-sh}}{s} - \frac{e^{-sh}}{s^2} + \frac{0 e^0}{s} + \frac{0 e^{-1}}{s^2} \right]$$

$$= \lim_{h \rightarrow \infty} \left[-\frac{h e^{-sh}}{s} + \frac{1}{s^2} \right]$$

$$\begin{aligned} h &\rightarrow \infty \\ e^{-\infty} &\rightarrow 0 \end{aligned}$$

$$= \lim_{h \rightarrow \infty} \left[-\frac{h}{s e^{sh}} + \frac{1}{s^2} \right]$$

$$\frac{1}{e^{\infty}} = \frac{1}{\infty}$$

limit

$$= \lim_{h \rightarrow \infty} \left[-\frac{1}{s^2 e^{sh}} + \frac{1}{s^2} \right] = 0 + \frac{1}{s^2} = \frac{1}{s^2}$$

$f(t)$

$$\mathcal{L}\{f(t)\} = F(s)$$

a

$$\frac{a}{s}$$

t^n

$$\frac{n!}{s^{n+1}}$$

e^{at}

$$\frac{1}{s-a}$$

$\sin(at)$

$$\frac{a}{s^2+a^2}$$

$\cos(at)$

$$\frac{s}{s^2+a^2}$$

$\sinh(at)$

$$\frac{a}{s^2-a^2}$$

$\cosh(at)$

$$\frac{s}{s^2-a^2}$$

Remark:

$$\mathcal{L}\{\alpha f(t) \pm \beta g(t)\}$$

$$= \alpha \mathcal{L}\{f(t)\} \pm \beta \mathcal{L}\{g(t)\}$$

α, β are constants

Ex:

Evaluate:

$$\textcircled{1} \mathcal{L}\{1\} = \frac{1}{s}$$

$$\textcircled{2} \mathcal{L}\{3t^2\} = 3 \mathcal{L}\{t^2\} = 3 \cdot \frac{2!}{s^{2+1}} = \frac{3 \cdot 2}{s^3} = \frac{6}{s^3}$$

$$\textcircled{3} \mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$$

$$\textcircled{4} \mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}$$

$$\textcircled{5} \mathcal{L}\left\{2 + 3t + \frac{1}{e^{-2t}} + 5 \sin(4t)\right\}$$

$$= \mathcal{L}\{2\} + 3 \mathcal{L}\{t\} + \mathcal{L}\{e^{2t}\} + 5 \mathcal{L}\{\sin(4t)\}$$

$$= \frac{2}{s} + \frac{3}{s^2} + \frac{1}{s-2} + 5 \cdot \frac{4}{s^2+16}$$

Remark :

$$\textcircled{1} \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$\textcircled{2} \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\textcircled{3} \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\textcircled{4} \sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$$

$$\textcircled{5} \cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$$

$$\textcircled{6} \sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

$$\textcircled{7} \cos(\alpha \pm \beta) = \cos \alpha \cos \beta \pm \sin \alpha \sin \beta$$

Ex:

Evaluate:

$$\textcircled{1} \mathcal{L} \{ \sin(3t) \cdot \sin(5t) \}$$

$$= \frac{1}{2} \mathcal{L} \{ \cos 2t - \cos 8t \}$$

$$= \frac{1}{2} \left[\frac{s}{s^2+4} - \frac{s}{s^2+64} \right]$$

$$\textcircled{2} \mathcal{L} \{ \cos(3t - \frac{\pi}{3}) \}$$

$$= \mathcal{L} \{ \cos(3t) \cdot \cos(\frac{\pi}{3}) + \sin(3t) \sin(\frac{\pi}{3}) \}$$

$$\frac{1}{2}$$

$$\frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} \mathcal{L} \{ \cos(3t) \} + \frac{\sqrt{3}}{2} \mathcal{L} \{ \sin(3t) \}$$

$$= \frac{1}{2} \cdot \frac{s}{s^2+9} + \frac{\sqrt{3}}{2} \cdot \frac{3}{s^2+9}$$

$$\textcircled{3} \mathcal{L} \{ \cos(3t + 2\pi) \}$$

$$= \mathcal{L} \{ \cos(3t) \} = \frac{s}{s^2 + 9}$$

$$\textcircled{4} \mathcal{L} \{ \sin(\frac{\pi}{2} - 4t) \}$$

$$= \mathcal{L} \{ \cos 4t \} = \frac{s}{s^2 + 16}$$

* Remark:

$$\mathcal{L} \{ e^{at} f(t) \} = \mathcal{L} \{ f(t) \} \Big|_{s \rightarrow s-a} = F(s-a)$$

Ex:

Evaluate

$$\textcircled{1} \mathcal{L} \{ e^{3t} \cos(5t) \}$$

$$= \mathcal{L} \{ \cos(5t) \} \Big|_{s \rightarrow s-3} = \frac{s}{s^2 + 25} \Big|_{s \rightarrow s-3} = \frac{s-3}{(s-3)^2 + 25}$$

$$② \mathcal{L} \{ e^{-2t} t^4 \}$$

$$= \mathcal{L} \{ t^4 \} \Big|_{s \rightarrow s+2} = \frac{4!}{s^5} \Big|_{s \rightarrow s+2} = \frac{24}{(s+2)^5}$$

Ex.:

Evaluate

$$\mathcal{L} \{ \sin^2(3t) \} = \mathcal{L} \left\{ \frac{1}{2} - \frac{1}{2} \cos(6t) \right\}$$

$$= \mathcal{L} \left\{ \frac{1}{2} \right\} - \frac{1}{2} \mathcal{L} \{ \cos(6t) \}$$

$$\frac{1}{2s} - \frac{1}{2} \cdot \frac{s}{s^2 + 36}$$

* Remark:

$$\mathcal{L} \{ t^n f(t) \} = (-1)^n \frac{d^n}{ds^n} [\mathcal{L} \{ f(t) \}]$$

* Ex 1

Evaluate

$$\textcircled{1} \mathcal{L} \{ t^2 \sin 3t \}$$

$$= (-1)^2 \cdot \frac{d^2}{ds^2} [\mathcal{L} \{ \sin 3t \}]$$

$$= \frac{d^2}{ds^2} \left[\frac{3}{s^2 + 9} \right] = \dots$$

$$\textcircled{2} \left[\frac{3}{s^2 + 9} \right]^2 = \frac{-6s}{(s^2 + 9)^2}$$

$$\textcircled{3} \left[\frac{3}{s^2 + 9} \right] = \frac{-6s \cdot (s^2 + 9)^2 + 6s \cdot 2(s^2 + 9) \cdot 2s}{(s^2 + 9)^4}$$

$$= \frac{-6s^2 - 54 + 24s^2}{(s^2 + 9)^3}$$

$$= \frac{18s^2 - 54}{(s^2 + 9)^3}$$

$$\textcircled{2} \mathcal{L} \{ t^{20} e^{3t} \} = \mathcal{L} \{ t^{20} \} /$$

$$s \rightarrow s-3$$

$$\frac{20!}{s^{21}} \bigg|_{s-3} = \frac{20!}{(s-3)^{21}}$$

$$\mathcal{L}(s) = \mathcal{L}\{f(t)\}$$

$$\mathcal{L}^{-1}\{\mathcal{L}(s)\} = f(t)$$

$$\begin{aligned} & \frac{a}{s} \\ & \frac{1}{s^n} \\ & \frac{1}{s-a} \\ & \frac{1}{s^2+a^2} \\ & \frac{s}{s^2+a^2} \\ & \frac{1}{s^2+a^2} \\ & \frac{s}{s^2-a^2} \end{aligned}$$

$$\begin{aligned} & a \\ & t^{n-1} e^{at} \frac{(n-1)!}{(n-1)!} \\ & \frac{\sin at}{a} \\ & \frac{\cos at}{a} \\ & \frac{\sinh at}{a} \\ & \frac{\cosh at}{a} \end{aligned}$$

Remark,

$$\mathcal{L}^{-1}\{\mathcal{L}(s-a)\} = e^{at} \mathcal{L}^{-1}\{f(s)\}$$

Ex:

Evaluate

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \frac{2}{s} \right\} = 2$$

$$\textcircled{2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} = \cos 3t$$

$$\textcircled{3} \mathcal{L}^{-1} \left\{ \frac{4}{s^2+20} \right\} = 4 \mathcal{L}^{-1} \left\{ \frac{1}{s^2+20} \right\}$$

$$= 4 \cdot \frac{\sin \sqrt{20} t}{\sqrt{20}} = \frac{2 \sin \sqrt{20} t}{\sqrt{5}}$$

$$\textcircled{4} \mathcal{L}^{-1} \left\{ \frac{3}{s-5} \right\} = 3 \mathcal{L}^{-1} \left\{ \frac{1}{s-5} \right\} = 3 e^{5t}$$

$$\textcircled{5} \mathcal{L}^{-1} \left\{ \frac{s-3}{(s+4)^2+36} \right\} = \mathcal{L}^{-1} \left\{ \frac{s+4-4+3}{(s+4)^2+36} \right\}$$

$$= e^{-4t} \cos(6t) - 7e^{-4t} \frac{\sin 6t}{6} = \mathcal{L}^{-1} \left\{ \frac{s+4}{(s+4)^2+36} \right\} - \frac{7}{(s+4)^2+36}$$

$$\textcircled{6} \mathcal{L}^{-1} \left\{ \frac{s-5}{(s-5)^2+49} \right\} = e^{5t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+49} \right\}$$

$$= e^{5t} \cdot \cos 7t$$

$$\textcircled{7} \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 4s - 5} \right\} = \mathcal{L}^{-1} \left\{ \frac{\frac{1}{2}}{s-1} - \frac{\frac{1}{2}}{s+5} \right\}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} - \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s+5} \right\}$$

$$= \frac{1}{2} e^t - \frac{1}{2} e^{-5t}$$

$$\frac{3}{s^2 + 4s - 5} = \frac{3}{(s-1)(s+5)} = \overset{\text{partial set}}{= \frac{A}{s-1} + \frac{B}{s+5}}$$

$$3 = A(s+5) + B(s-1)$$

$$\text{if } s = -5 \rightarrow 3 = A(0) + B(-5-1)$$

$$3 = -6B \quad (B = -\frac{1}{2})$$

$$\text{if } s = 1 \rightarrow 3 = A(1+5) + B(0)$$

$$3 = 6A \quad (A = \frac{1}{2})$$

$$3) \mathcal{L}^{-1} \left\{ \frac{5}{s^2 + 4s + 5} \right\}$$

$$\text{المخرج} \\ = s^2 + 4s + 5$$

$$= \mathcal{L}^{-1} \left\{ \frac{s + 2 - 2}{(s + 2)^2 + 1} \right\}$$

$$= \frac{s^2 + 4s + 4 - 4 + 5}{(s + 2)^2 + 1}$$

$$= \mathcal{L}^{-1} \left\{ \frac{s + 2}{(s + 2)^2 + 1} \right\} - 2 \mathcal{L}^{-1} \left\{ \frac{1}{(s + 2)^2 + 1} \right\}$$

$$= e^{-2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} - 2e^{-2t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\}$$

$$= e^{-2t} \cos t - 2e^{-2t} \sin t.$$

$$9) \mathcal{L}^{-1} \left\{ \frac{4}{(s^2 + 3)(s^2 + 5)} \right\} = 4 \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 3)(s^2 + 5)} \right\}$$

$$= \frac{4}{-2} \mathcal{L}^{-1} \left\{ \frac{(s^2 + 3) - (s^2 + 5)}{(s^2 + 3)(s^2 + 5)} \right\}$$

$$= -2 \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 5} - \frac{1}{s^2 + 3} \right\} = -2 \left[\frac{\sin \sqrt{5} t}{\sqrt{5}} - \frac{\sin \sqrt{3} t}{\sqrt{3}} \right]$$

17. w

$$\textcircled{10} \mathcal{L}^{-1} \left\{ \frac{-2}{(s^2+9)(s^2+4)} \right\}$$

$$\textcircled{11} \mathcal{L}^{-1} \left\{ \frac{s}{2s^2+4s+12} \right\}$$

$$\textcircled{12} \mathcal{L}^{-1} \left\{ \frac{s}{(2s-4)^2+98} \right\}$$

* Remark,

$$\mathcal{L}^{-1} \left\{ \mathcal{L}(f) \right\} = \frac{-1}{t} \mathcal{L}^{-1} \left\{ \mathcal{L}'(f) \right\}$$

$$\textcircled{13} \mathcal{L}^{-1} \left\{ \ln \left(\frac{s-3}{s+4} \right) \right\}$$

$$= \mathcal{L}^{-1} \left\{ \ln(s-3) - \ln(s+4) \right\} = \frac{-1}{t} \mathcal{L}^{-1} \left\{ \frac{1}{s-3} - \frac{1}{s+4} \right\}$$

$$= \frac{-1}{t} \mathcal{L}^{-1} \left\{ \frac{1}{s-3} - \frac{1}{s+4} \right\} = \frac{-1}{t} \left[e^{3t} - e^{-4t} \right]$$

$$(14) \mathcal{L}^{-1} \left\{ \frac{\pi}{2} \cdot \tan^{-1}(2s) \right\}$$

$$= \frac{1}{6} \mathcal{L}^{-1} \left\{ \left[\frac{\pi}{2} - \tan^{-1}(2s) \right] \right\}$$

$$= \frac{1}{6} \mathcal{L}^{-1} \left\{ 0 - \frac{2}{1+(2s)^2} \right\} = 4 \frac{1}{6} \mathcal{L}^{-1} \frac{2}{1+4s^2}$$

$$= \frac{2}{96} \mathcal{L}^{-1} \left\{ \frac{1}{\frac{1}{4} + s^2} \right\} = \frac{1}{26} \cdot \frac{\sin \frac{1}{2} t}{\frac{1}{2}}$$

$$= \frac{\sin(\frac{1}{2} t)}{6}$$

$$(15) \mathcal{L}^{-1} \left\{ \ln \left(\frac{s^2 + 4s + 4}{s^2 + 25} \right) \right\}$$

H.W

$$(16) \mathcal{L}^{-1} \left\{ \frac{1}{(s+4)^2} \right\} e^{-ub} \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\}$$

$$= e^{-ub} \frac{t^{19}}{19!}$$

Ex: Evaluate

$$\int_0^{\infty} \frac{\sin 5t}{e^{-2t}} dt$$

$$2 \int_0^{\infty} e^{2t} \sin 5t dt = \mathcal{L}^{-1} \left\{ \sin 5t \right\} \Big|_{s=2}$$

$$2 \frac{5}{s^2 + 2s} \Big|_{s=2} = 2 \cdot \frac{5}{(-2)^2 + 2s} = \frac{5}{29}$$

$$\mathcal{L} \{ \sin 5t \} = \int_0^{\infty} e^{-st} \sin(5t) dt$$

H.W.:

$$\text{Find } \mathcal{L}^{-1} \left\{ \frac{s-1}{(s^2-2s+4)^2} \right\}$$

$$\left(\frac{1}{s^2-2s+4} \right)' = \dots$$

Limit - step function

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

* Remark

$$\textcircled{1} \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$\textcircled{2} \mathcal{L}\{u(t-a) \cdot f(t)\} = e^{-as} \cdot \mathcal{L}\{f(t+a)\}$$

$$\textcircled{3} \mathcal{L}^{-1}\{e^{-as} \cdot f(s)\} = u(t-a) \cdot \mathcal{L}^{-1}\{f(s)\} \Big|_{t \rightarrow t+a}$$

Ex:

Evaluate

$$\textcircled{1} \mathcal{L}\{u(t-a)\} = \frac{e^{-4s}}{s}$$

$$\begin{aligned} \textcircled{2} \mathcal{L}\{u(t-5) \cdot 20\} &= \mathcal{L}\{u(t-5)\} \\ &= \frac{e^{-5s}}{s} \end{aligned}$$

$$\textcircled{3} \mathcal{L}\{4(t-4)t^2\} = e^{-4s} \mathcal{L}\{(t+4)^2\}$$

$$= e^{-4s} \mathcal{L}\{t^2 + 8t + 16\} = e^{-4s} \left[\frac{2!}{s^3} + \frac{8}{s^2} + \frac{16}{s} \right]$$

$$\textcircled{4} \mathcal{L}\{4(t-2)e^{3t}\} = \mathcal{L}\{4(t-2)\} \downarrow_{s \rightarrow s-3}$$

$$= \frac{e^{-2s}}{s} \downarrow_{s \rightarrow s-3}$$

$$= \frac{e^{-2(s-3)}}{s-3}$$

$$\textcircled{5} \mathcal{L}\{4(t-3) \cdot t \cdot e^{-5t}\}$$

$$= \mathcal{L}\{4(t-3) \cdot t\} \downarrow_{s \rightarrow s+5}$$

$$= \left[e^{-3s} \mathcal{L}\{t+3\} \right]_{s \rightarrow s+5}$$

$$= \left[e^{-3s} \left[\frac{1}{s} + \frac{3}{s} \right] \right]_{s \rightarrow s+5} = e^{-3(s+5)} \left[\frac{1}{(s+5)} + \frac{3}{s+5} \right]$$

$$\textcircled{6} \mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s} \right\} = 4(t-3)$$

$$\textcircled{7} \mathcal{L}^{-1} \left\{ \frac{2}{s^3(s-4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{e^{-4s}}{s^3} \right\}$$

$$= 24(t-4) \left[\mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\} \right]$$

$$= 24(t-4) \cdot \left[\frac{t^2}{2!} \right]_{t \rightarrow t-4}$$

$$= 24(t-4) \cdot \frac{(t-4)^2}{2}$$

$$\textcircled{8} \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{(s-4)^2 + 25} \right\}$$

$$= 4(t-2) \left[\mathcal{L}^{-1} \left\{ \frac{1}{(s-4)^2 + 25} \right\} \right]_{t \rightarrow t+2}$$

$$= 4(t-2) \left[e^{4t} : \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 25} \right\} \right]_{t \rightarrow t-2}$$

$$= 4(t-2) \cdot \left[e^{4t} \cdot \frac{\sin 5t}{5} \right]_{t \rightarrow t-2}$$

$$= 4(t-2) \cdot \frac{\sin 5(t-2)}{5}$$

$$\mathcal{L}^{-1} \left\{ \frac{3}{e^{3s} (s-5)^4} \right\}$$

$$= 3 \mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{(s-5)^4} \right\} = 3 u(t-3) \cdot \mathcal{L}^{-1} \left\{ \frac{1}{(s-5)^4} \right\}$$

$$= 3 u(t-3) \left[\mathcal{L}^{-1} \left\{ \frac{1}{s^4} \right\} \right]_{t \rightarrow t-3}$$

$$= 3 u(t-3) \left[\frac{e^{5t} t^{-3}}{3!} \right]_{t \rightarrow t-3}$$

$$= 3 u(t-3) \cdot e^{5(t-3)} \cdot \frac{(t-3)^3}{6}$$

Ex:

$$f(t) = \begin{cases} t & 0 < t < 2 \\ 3 & 2 < t < 5 \\ e^{-4t} & 5 < t \end{cases}$$

Find $\mathcal{L}\{f(t)\}$

$$g(t) = t + 4(t-2)(3-t) + 4(t-5)(e^{-4t} - 3)$$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{4(t-2)(3-t)\} + \mathcal{L}\{4(1-3)(e^{-4t} - 3)\}$$

$$= \frac{1}{s^2} + e^{-2s} \mathcal{L}\{3-1(t+2)\} + e^{-5s} \mathcal{L}\{e^{-4(t+5)} - 3\}$$

$$= \frac{1}{s^2} + e^{-2s} \mathcal{L}\{1-t\} + e^{-5s} \mathcal{L}\{e^{-20} - e^{-4t} - 3\}$$

$$= \frac{1}{s^2} + e^{-2s} \left[\frac{1}{s} - \frac{1}{s^2} \right] + e^{-5s} \left[\frac{e^{-20}}{s-4} - \frac{3}{s} \right]$$

~~*Remark,~~

$$\textcircled{1} \mathcal{L}\{y^{(n)}(t)\} = s^n \mathcal{L}\{y\} - s^{n-1} y(0) - \dots - y^{(n-1)}(0)$$

$$\textcircled{2} \quad n=1 \quad \mathcal{L}\{y'(t)\} = s \mathcal{L}\{y\} - y(0)$$

$$\textcircled{3} \quad n=2 \quad \mathcal{L}\{y''(t)\} = s^2 \mathcal{L}\{y\} - s y(0) - y'(0)$$

Ex,

Use Laplace transform to solve the following I.V.P

$$y'' - 4y' + 4y = 6e^{2b}, \quad y(0) = 0, \quad y'(0) = 0$$

$$\Rightarrow \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{6e^{2b}\}$$

$$= s^2 \mathcal{L}\{y\} - s y(0) - y'(0) - 4(s \mathcal{L}\{y\} - y(0)) + 4\mathcal{L}\{y\}$$

$$= \mathcal{L}\{6e^{2b}\}$$

$$b \rightarrow s-2$$

$$s^2 \mathcal{L}\{y\} - 4s \mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{6}{s-2}$$

$$\Rightarrow (s^2 - 4s + 4) \mathcal{L}\{y\} = \frac{6}{(s-2)^4}$$

$$\Rightarrow y(s) = \mathcal{L}\{y(b)\} = \frac{6}{(s^2 - 4s + 4)(s-2)^4}$$

no 0 ✓

$$y(b) = \mathcal{L}^{-1}\left\{\frac{6}{(s^2 - 4s + 4)(s-2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

$$= 6e^{2b} \mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} = 6e^{2b} \cdot \frac{b^5}{5!}$$